

Q 3.1 a)

In[245]:= $b[x, n, p] := n! / ((n - x)! * x!) * p^x * (1 - p)^{n - x}$

In[246]:= $b[x, n, p]$

$$\text{Out}[246]= \frac{(1 - p)^{n-x} p^x n!}{(n - x)! x!}$$

In[247]:= PowerExpand[Log[%]]

$$\text{Out}[247]= (n - x) \log[1 - p] + x \log[p] + \log[n!] - \log[(n - x)!] - \log[x!]$$

In[248]:= % /. {Log[n!] → n * Log[n] - n}

$$\text{Out}[248]= -n + n \log[n] + (n - x) \log[1 - p] + x \log[p] - \log[(n - x)!] - \log[x!]$$

In[249]:= % /. {Log[(n - x)!] → (n - x) * Log[(n - x)] - (n - x)}

$$\text{Out}[249]= -x + n \log[n] + (n - x) \log[1 - p] + x \log[p] - (n - x) \log[n - x] - \log[x!]$$

In[250]:= % /. {Log[n - x] → Log[n] - x/n}

$$\text{Out}[250]= -x + n \log[n] - (n - x) \left(-\frac{x}{n} + \log[n] \right) + (n - x) \log[1 - p] + x \log[p] - \log[x!]$$

In[251]:= % /. {Log[1 - p] → -p}

$$\text{Out}[251]= -p(n - x) - x + n \log[n] - (n - x) \left(-\frac{x}{n} + \log[n] \right) + x \log[p] - \log[x!]$$

In[252]:= Simplify[%]

$$\text{Out}[252]= -np + px - \cancel{\frac{x^2}{n}} + x \log[n] + x \log[p] - \log[x!]$$

$\times \log(np)$
Since $p \rightarrow 0$
 $n \rightarrow \infty$

In[253]:= Simplify[Exp[-np + px * (Log[n*p]) - Log[x!]]]

$$\text{Out}[253]= \frac{e^{-np} (np)^x}{x!} = \text{Poisson}(x, n, p) \quad QED$$

$$\begin{aligned}
 b) \langle (x)_m \rangle &= \sum_{x=0}^{\infty} \frac{e^{-N} N^x}{x!} (x(x-1)\cdots(x-m+1)) \\
 &= \sum \frac{e^{-N} N^x}{x!} \frac{x!}{(x-m)!} \\
 &= \sum \frac{e^{-N} N^x}{(x-m)!} \\
 &= Z^3
 \end{aligned}$$

c) Want $\langle x \rangle, \langle x^2 \rangle.$

$$\langle x \rangle = N \text{ using above result}$$

$$\begin{aligned}
 N^2 &= \langle x(x-1) \rangle = \langle x^2 - x \rangle \\
 &\quad = \langle x^2 \rangle - \langle x \rangle \text{ by linearity of expectations} \\
 \sigma^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
 &= \langle x^2 \rangle - \langle x \rangle + \langle x \rangle - \langle x \rangle^2 \\
 &= Z^2 + N - N^2
 \end{aligned}$$

$$\rightarrow \sigma = \sqrt{Z}$$

$$\begin{aligned}
 \therefore \langle x \rangle &= \frac{\sqrt{Z}}{Z} = \frac{-}{\sqrt{Z}}
 \end{aligned}$$

Q 3.2

$$\text{Error} = \text{RSD} = \frac{\sigma}{\langle x \rangle} = \frac{1}{\sqrt{N}}$$

for a Poisson process

$$0.01 = \frac{1}{\sqrt{N}} \Rightarrow N = 10^4 \text{ photons / s}$$

$$\frac{1}{10^6} = \frac{1}{\sqrt{N}} \Rightarrow N = 10^{12} \text{ photons / s}$$

$$E_{\text{photon}} = \frac{hc}{\lambda_{\text{visible}}} \quad \lambda_{\text{visible}} \approx 550 \text{ nm} = 5.5 \times 10^{-7} \text{ m}$$

$$10^4 \cdot E = 3.6 \cdot 10^{-15} \text{ W}$$

$$10^{12} \cdot E = 3.6 \cdot 10^{-7} \text{ W}$$

//

Q3.3 a) Johnson noise = $\langle V_{noise}^2 \rangle$

$$= 4kT R \Delta f \quad T = \cancel{R} \approx 293K$$
$$\Delta f = 20 \text{ kHz}$$
$$R = 10 \text{ k}\Omega$$
$$= 3.24 \mu\text{V}^2$$

$$\langle V_{RMS \text{ noise}} \rangle = 1.8 \mu\text{V} = 1.8 \times 10^{-6} \checkmark$$

$$20 \text{ dB} = 10 \log_{10} \left(\frac{\langle V_{signal}^2 \rangle}{\langle V_{noise}^2 \rangle} \right)$$

$$\rightarrow V_{RMS \text{ signal}} = 1.8 \times 10^{-5} \checkmark$$

b) 1 deg of thermal fluctuations for cap so

$$\frac{1}{2} C \langle V_{therm}^2 \rangle = \frac{1}{2} kT$$

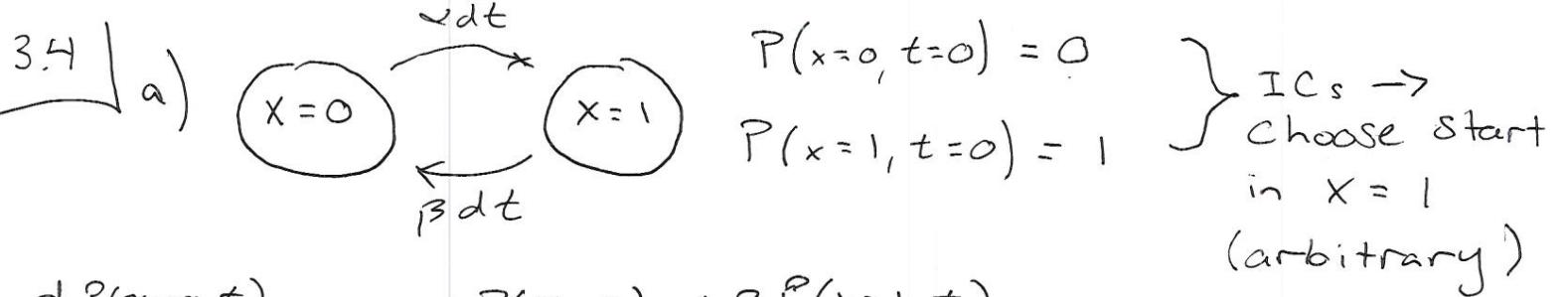
$$\text{If } \langle V_{therm}^2 \rangle = \langle V_{noise}^2 \rangle \quad C = \frac{kT}{(1.8 \mu\text{V})^2} = 1.25 \text{ nF}$$

c) $\langle I_{noise}^2 \rangle = 2q \langle I \rangle \Delta f$

$$0.01 = \frac{\langle I_{RMS \text{ noise}} \rangle}{\langle I \rangle} = \frac{\sqrt{2q \langle I \rangle \Delta f}}{\langle I \rangle}$$

$$= \frac{\sqrt{2q \Delta f}}{\langle I \rangle^{1/2}} \quad \rightarrow \quad \langle I \rangle = \left(\frac{\sqrt{2q \Delta f}}{0.01} \right)^2$$

$$= 6.4 \times 10^{-11} \text{ A}$$



$$\frac{d}{dt} P(x=0, t) = -\alpha P(x=0, t) + \beta P(x=1, t)$$

$$\frac{d}{dt} P(x=1, t) = \alpha P(x=0, t) - \beta P(x=1, t)$$

$$\text{or } \frac{d}{dt} \begin{pmatrix} P(x=0, t) \\ P(x=1, t) \end{pmatrix} = \begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \begin{pmatrix} P(x=0, t) \\ P(x=1, t) \end{pmatrix}$$

which we will call ~~$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t)$~~ $\frac{d}{dt} \vec{u}(t) = A \vec{u}(t)$

$$b) \det(A - \lambda I) = \det \begin{pmatrix} -\alpha - \lambda & \beta \\ \alpha & -\beta - \lambda \end{pmatrix} = 0$$

$$\rightarrow (-\alpha - \lambda)(-\beta - \lambda) - \alpha \beta = 0$$

$$\alpha \beta - \alpha \beta + (\alpha + \beta) \lambda + \lambda^2 = 0 \rightarrow$$

$$\boxed{\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = -(\alpha + \beta) \end{array}}$$

$$(A - \lambda_1 I) \vec{x}_1 = \vec{0} \quad \text{where } \vec{x}_1 \text{ is an eigenvector}$$

$$\begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix} \vec{x}_1 = \vec{0} \rightarrow \frac{\alpha}{\beta} x_1 = x_2$$

$$(A - \lambda_2 I) \vec{x}_2 = \vec{0}$$

$$\begin{pmatrix} \beta & \beta \\ \alpha & \alpha \end{pmatrix} \vec{x}_2 = \vec{0} \rightarrow -x_1 = x_2$$

$$\text{Pick } x_{1,0} = 1 \rightarrow \boxed{\vec{x}_1 = \begin{pmatrix} 1 \\ \alpha/\beta \end{pmatrix} \text{ & } \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Then eigenvector matrix $S = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha/\beta & -1 \end{pmatrix}$

S is invertible: $S^{-1} = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} & \frac{-\beta}{\alpha+\beta} \end{pmatrix}$

and $S^{-1}AS = \Delta = \begin{pmatrix} 0 & 0 \\ 0 & -(\alpha+\beta) \end{pmatrix}$

Now a differential equation has the form

$$\vec{u}(t) = c_1 e^{\lambda_1 t} \vec{x}_1 + c_2 e^{\lambda_2 t} \vec{x}_2 = S e^{\Delta t} \vec{c} \quad (\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix})$$

and $\vec{u}(0) = c_1 \vec{x}_1 + c_2 \vec{x}_2 = Sc$

or $\vec{c} = S^{-1} \vec{u}(0)$

So the complete expression for $\vec{u}(t)$ is:

$$\vec{u}(t) = S e^{\Delta t} S^{-1} \vec{u}(0)$$

and w/ our initial condition choice $\vec{u}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$\vec{u}(t) = \begin{pmatrix} 1 & 1 \\ \alpha/\beta & -1 \end{pmatrix} \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{-(\alpha+\beta)t} \end{pmatrix} \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} & \frac{-\beta}{\alpha+\beta} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{\beta}{\alpha+\beta} \begin{pmatrix} 1 - e^{-(\alpha+\beta)t} \\ \frac{\alpha}{\beta} + e^{-(\alpha+\beta)t} \end{pmatrix}$$

