

Q4.1 • $f(p) = p$ is continuous on $(-\infty, \infty)$

a) • $g(p) = \log p$ is continuous on $(0, \infty)$

• $p(x)$ is defined on $[0, 1]$

∴ $f(p)$ & $g(p)$ are continuous for all probabilities

• $f(p)g(p) = p \log p$ is continuous b/c product of continuous functions is continuous

• $-\sum p_i \log p_i$ is continuous b/c sum of continuous functions is continuous

∴ $H(p) = -\sum p_i \log p_i$ is continuous.

————— //

b) • $p(x)$ is defined on $[0, 1]$ ~~is continuous~~

• $0 \leq p \leq 1 \Rightarrow \log p \leq 0$

∴ $-p \log p \geq 0$

————— //

c) A function (here, $f(\vec{p}) = -\sum p_i \log p_i = H(\vec{p})$), subject to a single constraint (here, $g(\vec{p}) = \sum p_i = 1$), is described by N equations subject to a Lagrange Multiplier λ .

$$\text{i.e.: } \nabla f(\vec{p}) - \lambda \nabla g(\vec{p}) = 0$$

Here the K^{th} equation is

$$-\frac{\partial}{\partial p_K} \sum p_i \log p_i - \lambda \frac{\partial}{\partial p_K} \sum p_i = 0 \quad \text{or,}$$

$$-\log p_K - 1 = \lambda$$

Clearly p_K is constant and $p_K = p_i$ so

$$p_i = e^{-(1+\lambda)} \quad \text{and} \quad \sum e^{-(1+\lambda)} = 1 \Rightarrow$$

$$\lambda = \log N - 1 \quad \cancel{\text{where}} \quad \text{and}$$

$$p_i = e^{-\log N} = \frac{1}{N} \quad (\text{i.e. Uniform}(0, 1))$$

$$\therefore \text{Max}(H(p)) = -\sum \frac{1}{N} \log \left(\frac{1}{N} \right) = \underline{\underline{\log(N)}}$$

$$\begin{aligned}
 d) H(p, q) &= -\sum \sum p(x)q(y) \log(p(x)q(y)) \\
 &= -\sum \sum p(x)q(y) \log(p(x)) - \sum \sum p(x)q(y) \log(q(y)) \\
 &= -\sum p(x) \log(p(x)) - \sum q(y) \log(q(y)) \\
 &= H(p) + H(q)
 \end{aligned}$$

Q 4.2

$$\begin{aligned}
 a) I(x, y) &= \sum_x \sum_y p(x, y) \log \left(\frac{p(x, y)}{p(x)p(y)} \right) \\
 &= \sum_x \sum_y p(x, y) \log \left(\frac{p(x|y)p(y)}{p(x)p(y)} \right) \quad \text{via Bayes' rule} \\
 &= \sum_x \sum_y p(x, y) \log p(x|y) - \sum_x \sum_y p(x, y) \log p(\cancel{x}) \\
 &= \underline{H(x) - H(x|y)} \\
 &= \underline{H(y) - H(y|x)} \quad \text{by symmetry}
 \end{aligned}$$

$$\begin{aligned}
 b) \sum_x \sum_y p(x, y) \log(p(x, y)) - \sum \sum p(x, y) \log(p(x)p(y)) \\
 &= -H(x, y) - \sum_x p(x, y) \log(p(x)) - \sum_y p(x, y) \log(p(y)) \\
 &= -H(x, y) - \sum_x p(x) \log(p(x)) - \sum_y p(y) \log(p(y)) \\
 &= H(x) + H(y) - H(x, y)
 \end{aligned}$$

Q4.3 a) Given n lines with K faults

$$P_l(K \text{ faults}) = \binom{n}{K} \epsilon^K (1-\epsilon)^{n-K}$$

Given $n=3$ lines, error occurs if $K=2$ or $K=3$

$$P(\text{error}) = P_l(K=2) + P_l(K=3)$$

$$= \binom{3}{2} \epsilon^2 (1-\epsilon) + \epsilon^3$$

$$= 3\epsilon^2 - 3\epsilon^3 + \epsilon^3$$

$$= 3\epsilon^2 - 2\epsilon^3$$

$$\overbrace{\qquad\qquad\qquad}^{\epsilon} \qquad\qquad\qquad \overbrace{\qquad\qquad\qquad}^{K=2,3}$$

b) Let $\epsilon_0, \epsilon_1, \dots, \epsilon_N$ be the probabilities of an error on each successive layer of gates.

$$\text{Then } \epsilon_1 = 3\epsilon_0^2 - 2\epsilon_0^3 \text{ and } \underline{\epsilon_2 = 3\epsilon_1^2 - 2\epsilon_1^3}$$

$$\begin{aligned} c) \quad \epsilon_N &= 3\epsilon_{N-1}^2 - 2\epsilon_{N-1}^3 \quad \epsilon_N = 3\epsilon_{N-1}^2 - 2\epsilon_{N-1}^3 \\ &= 3 \left(\cancel{3\epsilon_0^2} \right) \quad = 3 \left(3^{N-1} \epsilon_0^{2(N-1)} + \dots \right)^2 \\ &\quad \cancel{- 2\epsilon_0^3} \end{aligned}$$

$$\begin{aligned} \epsilon_N &= 3\epsilon_{N-1}^2 - 2\epsilon_{N-1}^3 \\ &= 3 \left(3^{2(N-2)} \epsilon_0^{2(N-1)} + \dots \right)^2 - 2 \left(3^{3(N-2)} \epsilon_0^{3(N-1)} + \dots \right)^3 \\ &= O(\epsilon_0^{2N}) \quad \text{w/ } 3^N \text{ bits} \end{aligned}$$

$$Q 4.4 \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\ln(p(x)) = -\ln(\sqrt{2\pi\sigma^2}) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$= \int_{-\infty}^{\infty} p(x) \ln(p(x)) dx$$

$$= \ln(\sqrt{2\pi\sigma^2}) \int_{-\infty}^{\infty} p(x) dx + \int_{-\infty}^{\infty} p(x) \frac{(x-\mu)^2}{2\sigma^2} dx$$

$$= \ln(\sqrt{2\pi\sigma^2}) + \frac{\langle (x-\mu)^2 \rangle}{2\sigma^2}$$

$$= \ln(\sqrt{2\pi\sigma^2}) + \frac{\langle (x-\mu)^2 \rangle}{2 \langle (x-\mu)^2 \rangle}$$

$$= \frac{1}{2} \ln(2\pi\sigma^2) + \frac{1}{2}$$

$$= \frac{1}{2} \ln(2\pi\sigma^2) + \frac{1}{2} \ln(e)$$

$$= \frac{1}{2} \ln(2\pi e\sigma^2) \text{ nats}$$

$$= \underline{\frac{1}{2} \ln(2\pi e\sigma^2) \text{ nats}}$$

$$= \frac{1}{2} \log_2(2\pi e\sigma^2) \cdot \frac{1}{\log_2(e)} \text{ nats} \cdot \log_2(e) \frac{\text{bits}}{\text{nat}}$$

$$= \underline{\frac{1}{2} \log_2(2\pi e\sigma^2) \text{ bits}}$$

Q 4.5

a) $C = \Delta f \log \left(1 + \frac{S}{N} \right)$

$$= 3300 \text{ Hz} \log_2 (1 + 100)$$

$$= 21972 \text{ bits/s}$$

b) $C = 10^9 = 3300 \text{ Hz} \cdot \log_2 \left(1 + \frac{S}{N} \right)$

$$\frac{S}{N} = 2^{10^9 / 3300} - 1$$

• 11. ~~2~~ $2^{3 \cdot 10^5}$

• 11. $10^{9 \cdot 10^4}$

• 11. 10^{10^5}

= 10^6 dB

$$\boxed{Q4.6} \quad \text{Bias} = \langle f \rangle - x_0$$

$$\langle f \rangle = \left\langle \frac{1}{n} \sum x_i \right\rangle$$

$$= \frac{1}{n} \sum \langle x_i \rangle$$

$= x_0$ i.e. it is an unbiased estimator
of the mean

Cramér-Rao inequality: $\sigma^2(f) \geq \frac{1}{J_N(x)}$
(here, $x = x_0$)

$$\sigma^2(f) = \langle (f - x_0)^2 \rangle$$

$$= \left\langle \left(\frac{1}{n} \sum x_i - x_0 \right)^2 \right\rangle$$

$$= \frac{1}{n^2} \sum \langle (x_i - x_0)^2 \rangle$$

$$= \sigma^2 / n$$

$$J_N(x_0) = N J(x_0)$$

$$= n \left\langle \left(\frac{2}{2x_0} \log \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[-\frac{(x-x_0)^2}{2\sigma^2} \right] \right) \right)^2 \right\rangle$$

$$= n \left\langle \left(\frac{2}{2x_0} \left(-\frac{(x-x_0)^2}{2\sigma^2} \right) \right)^2 \right\rangle$$

$$= n \left\langle \left(\frac{x-x_0}{\sigma^2} \right)^2 \right\rangle$$

$$= n \frac{\sigma^2}{\sigma^4} = \frac{n}{\sigma^2}$$

$\therefore \sigma^2(f) = \frac{1}{J_N(x_0)}$ and achieves the bound