

Q 6.1 Prove $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k \quad (6.4)$$

$$= \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m \quad \textcircled{1} \quad (6.7)$$

$$\delta_{il}\delta_{jm} A_j B_l C_m = \begin{cases} \delta_{il} A_j B_l C_j & \text{if } j=m \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Similarly } -\delta_{im}\delta_{jl} A_j B_l C_m = \begin{cases} -\delta_{im} A_j B_j C_m & \text{if } j=l \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \textcircled{1} &= \delta_{il} A_j B_l C_j - \delta_{im} A_j B_j C_m \\ &= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B}) \end{aligned}$$

$$\therefore \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} \cdot (\vec{A} \cdot \vec{B}) \quad \underline{\text{QED}}$$

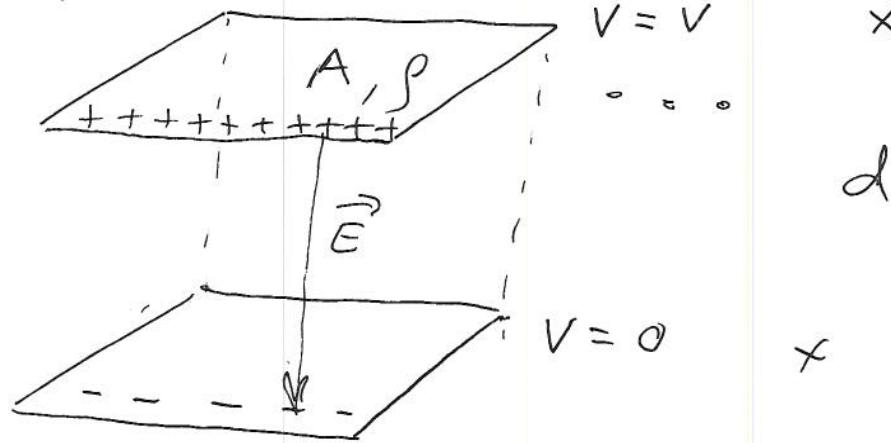
$$\text{If } \vec{A} = \vec{B} = \nabla, \vec{C} = \vec{E}$$

$$\begin{aligned} \nabla \times (\nabla \times \vec{E}) &= \nabla \cdot (\nabla \cdot \vec{E}) - \vec{E} (\nabla \cdot \nabla) \\ &= \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \quad \text{as required} \end{aligned}$$

Q 6.2

= (infinite)

a)



Gauss' Law: $\nabla \cdot \vec{D} = \rho$, $\int_V \nabla \cdot \vec{D} dV = \int_V \rho dV$

Divergence Theorem:

$$\int_V \nabla \cdot \vec{D} dV = \int_S \vec{D} \cdot d\vec{A}$$

$$S \circ \int_S \vec{D} \cdot d\vec{A} = \int_V \rho dV$$

$$\epsilon E A = Q \rightarrow E = Q / \epsilon A$$

$$V = - \int \vec{E} \cdot d\vec{l} = - Ed \Big|_{d=d}^{d=0} = Ed$$

$$\therefore V = Qd / \epsilon A$$

$$\text{Now } C \equiv Q / V \quad \therefore$$

$$C = \frac{\epsilon A}{d}$$

b) Internal displacement current = $\int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A}$

$$= \int_S \epsilon \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

$$= \int_S \frac{\epsilon}{d} \frac{\partial V}{\partial t} \cdot d\vec{A} = \frac{\epsilon A}{d} \dot{V} = C \dot{V} = I \quad \text{as required}$$

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c) Let U = energy density
 S = stored energy

$$S = \int_V U dV$$

$$= \frac{1}{2} \int_V \vec{E} \cdot \vec{D} dV$$

$$= \frac{1}{2} \epsilon \int_V E^2 dV$$

$$= \frac{\epsilon}{2} \cdot E^2 \cdot A \cdot d$$

$$= \frac{\epsilon}{2} \cdot \frac{Q^2}{\epsilon^2 A^2} \cdot A \cdot d$$

$$= \frac{1}{2} \frac{Q^2 \cdot d}{\epsilon A}$$

$$= \frac{1}{2} C^2 V^2 \cdot \frac{1}{C}$$

$$= \frac{1}{2} CV^2$$

~~\neq~~

$$\text{d) Total energy} = 10 \text{ V} \cdot 10 \text{ A} \cdot \text{h}$$
$$= 100 \text{ W} \cdot 3600 \text{ s}$$
$$= 3.6 \times 10^5 \text{ J}$$

$$S = 3.6 \times 10^5 \text{ J} = \frac{1}{2} CV^2$$
$$= \frac{1}{2} C (10 \text{ V})^2$$

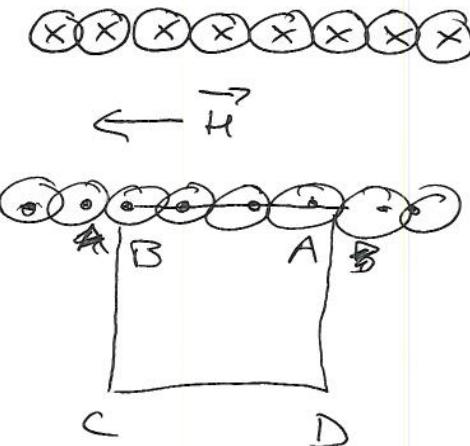
$$\rightarrow C = \underline{\underline{7.2 \text{ kF}}}$$

$$A = \frac{d \cdot C}{\epsilon_0} = \underline{\underline{8.1 \times 10^8 \text{ m}^2}}$$

$$\frac{A}{(0.1 \text{ m})^2} \times 10^{-6} \text{ m} = \underline{\underline{8.1 \times 10^4 \text{ m tall stack}}}$$

Q 6.3

a) Suppose



$$\oint \vec{H} \cdot d\vec{l} = \int_B^A \vec{H} \cdot d\vec{l} + \int_C^B \vec{H} \cdot d\vec{l} + \int_C^D \vec{H} \cdot d\vec{l} + \int_A^D \vec{H} \cdot d\vec{l}$$

$$= H l$$

$$= \int_S \vec{J} \cdot d\vec{A}$$

$$= n I \cdot l$$

$$\rightarrow Hl = nIl \quad \text{or} \quad \boxed{\mu = nI}$$

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$$b) U = \frac{1}{2} \vec{B} \cdot \vec{H}$$

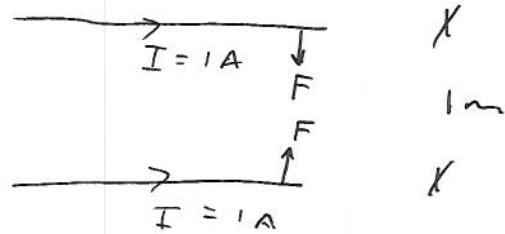
$$S = \frac{1}{2} \int_V \vec{B} \cdot \vec{H} \cdot dV$$

$$= \frac{1}{2} \int \mu H^2 dV$$

$$= \boxed{\frac{1}{2} \mu \cdot n^2 I^2 \cdot \pi r^2 l}$$

$$\begin{aligned} c) F &= \nabla S = \frac{\partial}{\partial r} \frac{1}{2} \mu_0 n^2 I^2 \pi r^2 l \\ &= \mu_0 n^2 I^2 \pi r l \\ &= \frac{\pi r l B^2}{\mu_0} \\ &= \pi \cdot 0.5 \text{ m} \cdot 2 \text{ m} \cdot (10 \text{ T})^2 / \mu_0 \\ &= 2.5 \times 10^8 \text{ N} \end{aligned}$$

Q 6.4]



$$\begin{aligned} d\vec{F} &= I d\vec{l} \times \vec{B} \\ &= I \mu_0 \cdot d\vec{l} \times \vec{H} \\ &= \frac{I^2 \mu_0}{2\pi r} dl \end{aligned}$$

For a 1m segment :

$$\begin{aligned} F &= \mu_0 \cdot (1\text{A})^2 \cdot 1\text{m} / 2\pi \text{m} \\ &= 2 \times 10^{-7} \text{ N} \end{aligned}$$

$$Q6.5) \text{a) Flux density} = \vec{P}(t) = \vec{E}(t) \times \vec{H}(t)$$

$$|\vec{P}(t)| = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2(t)$$

$$= \vec{E}_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)} \cdot \sqrt{\frac{\epsilon_0}{\mu_0}}$$

$$\text{Time average: } \langle |\vec{P}(t)| \rangle$$

$$= \text{Re} \left\{ \sum_{n=1}^{\infty} \int_{-\pi/\omega}^{\pi/\omega} |\vec{P}(t)| dt \right\}$$

$$= \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2$$

$$\text{Now } W = I \cdot K_W = \int_S \vec{P} \cdot d\vec{A}$$

$$= \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 \cdot m^2$$

$$E_0^2 = \frac{2 K_W}{\sqrt{\frac{\epsilon_0}{\mu_0}} \cdot m^2}$$

$$E_0 = 8.68 \text{ V/cm}$$