MAS 862 P.I.T.

Problem Set Chapter 3

(3.1)(α) Starting from the binomial distribution

we have : $p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$ (1)

We need to show that $(1) = \frac{e^{-np}(np)^x}{x!}$ [for large n and small x]

or $\frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x} = \frac{e^{-np}(np)^x}{x!}$

or $\frac{n!}{n-x!} (1-p)^{n-x} = e^{-np} n^x \overset{\ln}{\Longleftrightarrow} \ln n! - \ln(n-x)! + (n-x)\ln(1-p) = -np + x\ln n$

Using $\ln(1+x) \simeq x$, Stirling's formula and $n-x \simeq n$:

$$\cancel{n\ln n - n - (n-x)\ln(n-x)} + (n-x)(-p) = -np + x\ln n$$

or

$$x\ln n - np = -np + x\ln n$$

which is true. □

(β) $$\langle x(x-1)\cdots(x-m+1)\rangle = \int x(x-1)\cdots(x-m+1)\,p(x)\,dx$$

$$= \sum_{x=0}^{\infty} x(x-1)\cdots(x-m+1)\frac{e^{-N}N^x}{x!}\,dx$$

$$= \sum_{x=m}^{\infty} \frac{e^{-N}N^x}{(x-m)!}\,dx$$

$$= N^m \sum_{x=m}^{\infty} \frac{e^{-N}N^{x-m}}{x-m!}\,dx = N^m$$

as $\sum_{x=0}^{\infty} \frac{e^{-N}N^x}{x!} = 1$ by definition. □

(c) for $m=1$ $\langle x\rangle = N$

for $m=2$ $\langle x(x-1)\rangle = N^2 \Rightarrow \langle x^2 - x\rangle = N^2 \Rightarrow \langle x^2\rangle = N^2 + \langle x\rangle$

we have $\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = N^2 + \langle x\rangle - N^2 = \langle x\rangle = N$

$\Rightarrow \sigma = \sqrt{N}$

thus $\frac{\sigma}{\langle x\rangle} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$

$\square$

(3.2) We model the distribution of incoming photons with a Poisson distribution with average rate N.

thus $p(x) = \frac{e^{-N} N^x}{x!}$.

In that case the average rate can be estimated by (3.19)

So if we want 1% accuracy then we have

$0.01 \propto \frac{1}{\sqrt{N}}$ or $\underline{N \propto 10^4}$ photons/second

for 0.001‰ using the same equation we have $\underline{N \propto 10^{12}}$ photons/second

- Assuming a green photon, $\lambda = 532\,nm$

then the energy per photon is $E = hf = h\frac{c}{\lambda} = 6.62\cdot 10^{-34}\,J\cdot s \cdot 3\times 10^8\,ms^{-1} \cdot \frac{1}{532\,nm}$

which gives $E = 3.7\times 10^{-19}\,J/\text{photon}$

So for 1% $\rightarrow E = 3.7\times 10^{-19}\,J/\text{photon} \times 10^4\,\text{photons}/s = 3.7\times 10^{-15}$ Watt

for 0.001‰ $\rightarrow E = 3.7\times 10^{-19}\,J/\text{photon} \times 10^{12}\,\text{photons}/s = 3.7\times 10^{-7}$ Watt

(3.3) (a) $\Delta f = 20\,kHz$, $R = 10\,k\Omega$, $T = 300\,K$ (room temperature)

From (3.34) $\langle V_{noise}^2 \rangle = 4kTR\Delta f = 4 \cdot 1.38\times10^{-23}\,J/K \cdot 300\,K \cdot 10^4\,\Omega \cdot 2\cdot10^4\,s^{-1}$

$$= 3.31\times10^{-12}\,V^2$$

$$V_{noise}^{RMS} = \sqrt{\langle V_{noise}^2\rangle} = 1.81\times10^{-6}\,V$$

So we want $20\log\frac{V_{input}^{RMS}}{V_{noise}^{RMS}} = 20\,dB$ or $V_{input}^{RMS} = 10\,V_{noise}^{RMS}$

So we must have $V_{input}^{RMS} = 1.81\times10^{-5}\,V$

(b) A capacitor is a single degree of freedom, so from the equipartition theorem we have:

$$\tfrac{1}{2}C\langle V_{noise}^2\rangle = \tfrac{1}{2}kT$$

or $$C = \frac{kT}{\langle V_{noise}^2\rangle} = \frac{1.38\times10^{-23}\,J/K \times 300\,K}{3.31\times10^{-12}\,V^2} = 1.25\,nF$$

(c) From eq (3.33) we have that: $\langle I_{noise}^2\rangle = 2q\langle I\rangle\Delta f$

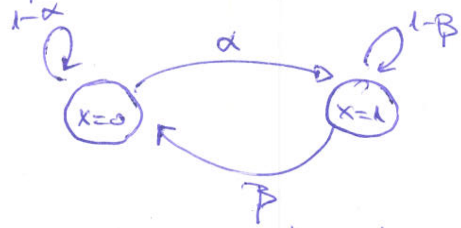
or $I_{noise}^{RMS} = \sqrt{2q\Delta f}\,\langle I\rangle^{1/2} = \sqrt{2\cdot1.6\times10^{-19}\times2\cdot10^4}\,\langle I\rangle^{1/2} = 8.19\times10^{-7}\langle I\rangle^{1/2}$ (1)

we want $I_{noise}^{RMS} = 0.01\langle I\rangle \xrightarrow{(1)} 8.19\times10^{-7}\langle I\rangle^{1/2} = 10^{-2}\langle I\rangle$

or $\langle I\rangle^{1/2} = 8.19\times10^{-5}$

or $\langle I\rangle = 6.7\times10^{-11}\,A$

(3.4) (a) Assuming the system has no memory, I can model it with a first order Markov process like this:



Assuming at time t we have p_0^t, p_1^t with $p_0+p_1=1$ the probabilities to be at each state, then after dt we have:

$$\begin{bmatrix} p_0^{(t+dt)} \\ p_1^{(t+dt)} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} p_0^{(t)} \\ p_1^{(t)} \end{bmatrix} \quad (1)$$

then $$\begin{bmatrix} \frac{dp_0}{dt} \\ \frac{dp_1}{dt} \end{bmatrix} \quad \begin{bmatrix} p_0^{(t+dt)} \\ p_1^{(t+dt)} \end{bmatrix} - \begin{bmatrix} p_0^{(t)} \\ p_1^{(t)} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} p_0^{(t)} \\ p_1^{(t)} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_0^{(t)} \\ p_1^{(t)} \end{bmatrix}$$

or $$\begin{bmatrix} \frac{dp_0}{dt} \\ \frac{dp_1}{dt} \end{bmatrix} = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \quad (2)$$

Also the stationary probabilities ($t \to \infty$) are found by

$$\begin{bmatrix} p_0 & p_1 \end{bmatrix} \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \quad \text{which give: } \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{bmatrix} \quad (3)$$

(β) If $\vec{p}(t) = \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix}$ then by (2) we have: $\dot{\vec{p}}(t) = A\,\vec{p}(t)$

To solve the differential equation we must find the eigenvalues and eigen vectors of A.

We have $|A-\lambda I| = 0$ or $(-\alpha-\lambda)(-\beta-\lambda) - \alpha\beta = 0$ or $\lambda[\lambda-(\alpha+\beta)] = 0$ which solved gives us eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -(\alpha+\beta)$ with eigenvectors $u_1 = \begin{pmatrix} 1 \\ \alpha/\beta \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

then the solutions are given by

$$\vec{p}(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 \quad \text{or} \quad \vec{p}(t) = c_1 \begin{pmatrix} 1 \\ \alpha/\beta \end{pmatrix} + c_2 e^{-(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{or} \quad \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{-(\alpha+\beta)t} \\ c_1 \alpha/\beta - c_2 e^{-(\alpha+\beta)t} \end{bmatrix} \quad ④$$

for ease of numbers

Assuming we start at state 1 $\Rightarrow \begin{bmatrix} p_0(0) \\ p_1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then for $t=0$, ④ $\Rightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 \alpha/\beta - c_2 = 1 \end{cases} \rightarrow \begin{array}{l} c_1 = \dfrac{1}{1+\alpha/\beta} \\ c_2 = \dfrac{-1}{1+\alpha/\beta} \end{array}$

Plugging those back to ④ we have

$$\begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = \frac{1}{1+\frac{\alpha}{\beta}} \begin{bmatrix} 1 - \alpha/\beta\, e^{-(\alpha+\beta)t} \\ \alpha/\beta + e^{-(\alpha+\beta)t} \end{bmatrix}$$

(c) $$\langle x_t x_{t+\tau} \rangle = \sum_{x_t=0,1} \sum_{x_{t+\tau}=0,1} x_t x_{t+\tau} p(x_t, x_{t+\tau})$$

$$\overset{\text{non zero terms}}{=} 1 \cdot 1 \cdot p(x_t = 1, x_{t+\tau} = 1)$$

$$= p(x_{t+\tau} = 1 \mid x_t = 1) \cdot p(x_t = 1) \quad ⑤$$

Assuming $t \to \infty$, $p(x_t = 1) = \dfrac{\alpha}{\alpha+\beta}$ and $p(x_{t+\tau}=1 \mid x_t = 1) = p_1(\tau) = \dfrac{1}{1+\alpha/\beta}\left(\frac{\alpha}{\beta} + e^{-(\alpha+\beta)\tau}\right)$

thus ⑤ $\Leftrightarrow \langle x_t x_{t+\tau} \rangle = \dfrac{\alpha\beta}{(\alpha+\beta)^2}\left(\dfrac{\alpha}{\beta} + e^{-(\alpha+\beta)\tau}\right)$

(d) From Wiener-Khichin Theorem we have $\langle x_t x_{t+\tau} \rangle = \displaystyle\int_{-\infty}^{\infty} S(f) e^{-i2\pi f \tau} df$

thus, taking the inverse $S(f) = \displaystyle\int_{-\infty}^{\infty} \langle x_t x_{t+\tau} \rangle e^{i2\pi f\tau} d\tau$ ...

which gives $S(f) = \int_{-\infty}^{\infty} \frac{\alpha\beta}{(\alpha+\beta)^2}\left(\alpha/\beta + e^{-(\alpha+\beta)\tau}\right) e^{i2\pi f\tau}\, d\tau$

$$= \int_{-\infty}^{\infty} \frac{\alpha^2}{(\alpha+\beta)^2} e^{i2\pi f\tau}\, d\tau + \int_{-\infty}^{\infty} \frac{\alpha\beta}{(\alpha+\beta)^2} e^{-\tau(\alpha+\beta-i2\pi f)}\, d\tau$$

$\mathcal{F}(e^{i2\pi f\tau}) = \delta(f)$

$$= \left(\frac{\alpha}{\alpha+\beta}\right)^2 \delta(f) + \frac{\alpha\beta}{\alpha+\beta^2}\, \mathcal{F}\left(e^{-\tau(\alpha+\beta)}\right) \quad (6)$$

Lorentzian

(e) We want $S(f) = 1/2\, S(0) \overset{(6)}{\Rightarrow} S(f) = 1/2\left(\frac{\alpha^2}{\alpha+\beta}\right) \Rightarrow \dots$