MAS 862 P. I. T.

Problem Set Chapter 4

(4.1). To show continuity for a function $f(x)$ we have to find $\delta > 0$ such that for any $\varepsilon > 0$ and $\forall x_0 \in D_f$

$$\text{if} \quad |x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \varepsilon$$

It suffices to show that $\lim_{x \to x_0} |f(x) - f(x_0)| = 0$

for $f(x) = x \log x$: $\lim_{x \to x_0} |x \log x - x_0 \log x_0| = 0$ $\square$

↳ sum of continuous = continuous

- We have $0 \le p \le 1 \Rightarrow \log p \le 0$

  thus $-p \log p \ge 0 \quad \forall \, p \in [0, 1]$

  and thus $H(p) = -\sum_i p_i \log p_i \ge 0$

- From non-negativity $\min H(p) = 0$

  To find max we have to solve the optimization problem $\max H(p_1, \ldots, p_N)$ s.t. $\sum_{i=1}^{N} p_i = 1$. (1)

  Using Lagrange multipliers we have:

$$\frac{\partial H}{\partial p_j} - \lambda \frac{\partial \sum_i p_i}{\partial p_j} = 0$$

$$\text{or} \quad -\frac{\partial}{\partial p_j} \left(\sum_i p_i \log p_i - \lambda p_i \right) = 0$$

$$\text{or} \quad -\log p_j - 1 - \lambda = 0$$

$$\text{or} \quad p_j = e^{-(1+\lambda)}$$

Substituting to (1): $\sum_{i=1}^{N} e^{-(1+\lambda)} = 1 \rightarrow \lambda = \log N - 1$ $\Bigg\} \Rightarrow p_j = e^{-\log N} = \frac{1}{N}$

thus $\max H(p) = \sum_{i=1}^{N} \frac{1}{N} \log \frac{1}{N} = \log N$. So $H(p) \in [0, \log N]$

$$\bullet \quad H(p,q) = -\sum_x \sum_y p(x)q(y) \log[p(x)q(y)]$$

$$= -\sum_x \sum_y p(x)q(y) \log p(x) - \sum_x \sum_y p(x)q(y) \log q(y)$$

$$= -\underbrace{\sum_y q(y)}_{1} \underbrace{\sum_x p(x) \log p(x)}_{H(p)} - \underbrace{\sum_x p(x)}_{1} \underbrace{\sum_y q(y) \log q(y)}_{H(q)}$$

$$= H(p) + H(q) \qquad \square$$

(4.2) $$I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

$$= \sum_x \sum_y p(x,y) \log p(x,y) - \sum_x \overbrace{\sum_y p(x,y)}^{p(x)} \log p(x) - \sum_y \overbrace{\sum_x p(x,y)}^{p(y)} \log p(y)$$

$$= \sum_x \sum_y p(x,y) \log p(x,y) - \sum_x p(x) \log p(x) - \sum_y p(y) \log p(y)$$

$$= -H(x,y) + H(x) + H(y) \qquad \square$$

$$I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

likewise $p(y)p(x|y)$ for last equation

$$= \sum_x \sum_y p(x,y) \log \frac{p(x)p(y|x)}{p(x)\ p(y)}$$

$$= \sum_x \sum_y p(x,y) \log p(y|x) - \sum_y \overbrace{\sum_x p(x,y)}^{p(y)} \log p(y)$$

$$= -H(y|x) + H(y) \qquad \square$$

(4.4) Gaussian process: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-x_0)}{2\sigma^2}}$

So: $H(p(x)) = -\int_{-\infty}^{\infty} p(x) \log p(x) dx = -\langle \log p(x) \rangle = -\left\langle \ln \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{(x-x_0)^2}{2\sigma^2} \right\rangle$

$$= \ln \sqrt{2\pi\sigma^2} + \frac{\overbrace{\langle x-x_0 \rangle^2}^{\sigma^2}}{2\sigma^2} = \frac{1}{2} \ln 2\pi\sigma^2 + \frac{1}{2} = \frac{1}{2} \ln(2\pi e\sigma^2)$$

(4.4)(a) Capacity: $C = \Delta f \log\left(1+\frac{S}{N}\right) = 3300\, s^{-1} \log\left(1+10^{20/10}\right)$

$= 22$ kbps

(b) We want $C = 1$ Gbps

or $10^9 = 3300 \log\left(1+\frac{S}{N}\right)$

or $\log\left(\frac{S}{N}+1\right) = \frac{10^9}{3300}$ or $\frac{S}{N} = 2^{\frac{10^9}{3300}} - 1$

which gives $\frac{S}{N} \approx 2^{3\times10^5}$

$2^3 \simeq 10$

thus $SNR = 10 \log_{10} 2^{3\times10^5} \simeq 10 \log_{10} 10^{10^5} = 10^6$ dB

(4.3)(a) $P_e = \varepsilon$.

For the majority voting to produce an error we need two out of three bits incorrect.

By setting P_e as the success probability in a Binomial distribution we have $P(2) = \frac{3!}{2!1!} P_e^2 (1-P_e)^2 = 1.5 \cdot \epsilon^2 (1-\epsilon^2)$

(b) Thinking the same way, now $P_\epsilon' = P(2)$

and thus $P'(2) = \frac{3!}{2!1!} P^2(2)(1-P(2))^2 \underset{\epsilon\to 0}{\approx} \frac{3}{2}(1.5\epsilon^2)^2(1-(1.5\epsilon^2)^2) \simeq 1.5(1.5\epsilon^2)$

(c) Applying the same process N times gives us

$$P_\epsilon^N \approx 1.5 \times (1.5\epsilon)^{2N} = \frac{3}{2} \times \left(\frac{3}{2}\epsilon\right)^{2N}$$

we require 3^N bits.

If $\epsilon > 2/3$, then probability of error increases exponentially

If $\epsilon < 2/3$ —— " —— decreases —— " ——

(4.6) We have to check for bias and variance.

$$\circ \langle f \rangle = \left\langle \frac{1}{n}\sum_{i=1}^{n} x_i \right\rangle = \frac{1}{n}\sum_{i=1}^{n}\langle x_i\rangle = \frac{1}{n}\sum_{i=1}^{n} x_0 = x_0$$

$$\circ \langle f - x_0\rangle^2 = \left\langle \left[\frac{1}{n}\sum_{i=1}^{n} x_i - x_0\right]\right\rangle^2 = \left\langle \left(\frac{1}{n}\sum_{i=1}^{n} x_i - x_0\right)\left(\frac{1}{n}\sum_{k=1}^{n} x_k - x_0\right)\right\rangle$$

$$= \left\langle \frac{1}{n}\sum_{i=1}^{n}(x_i - x_0)\right\rangle \left\langle \frac{1}{n}\sum_{k=1}^{n}(x_k - x_0)\right\rangle$$

$$= \frac{1}{n}\sum_{i=1}^{n}\langle (x_i - x_0)\rangle \times \frac{1}{n}\sum_{k=1}^{n}\langle (x_k - x_0)\rangle$$

assuming variables uncorellated

$$= \frac{1}{n^2}\sum_{i=1}^{n}\langle (x_i - x_0)\rangle^2$$

$$= \frac{\sigma^2}{n^2}$$

now we can calculate fisher information

i.e. $$J_n(x_0) = n J(x_0) = n\left\langle \left[\partial_{x_0} \log N(x_0, \sigma^2)\right]^2\right\rangle$$

$$= n\left\langle \left[\partial_{x_0}\left[\log\frac{1}{\sqrt{2\pi\sigma^2}} + \frac{(x-x_0)^2}{2\sigma^2}\right]\right]^2\right\rangle$$

$$= n\left\langle \left[\frac{x-x_0}{\sigma^2}\right]^2\right\rangle$$

$$= n\frac{\sigma^2}{\sigma^4} = \frac{n}{\sigma^2} \equiv \text{Cramér-Rao lower bound}$$

$\square$