MAS.862 P.I.T.
Problem Set Chapter 8

(8.1) We have a vector potential $\vec{A}(r,\vartheta) = \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r}(\cos\vartheta\,\hat{r} - \sin\vartheta\,\hat{\vartheta})$

To find the electrical field, I will use eq. (8-21): $\vec{E} = \frac{\nabla\cdot(\nabla\cdot\vec{A})}{i\omega\mu_0\varepsilon_0} - i\omega\vec{A}$

In spherical coordinates we have:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(A_\vartheta \sin\vartheta)}{\partial\vartheta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\mu_0 I_0 d}{4\pi}\cos\vartheta\, r\, e^{-ikr}\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(-\frac{\mu_0 I_0 d}{4\pi r}e^{-ikr}\sin^2\vartheta\right)$$

$$= \frac{\mu_0 I_0 d}{4\pi r^2}e^{-ikr}\cos\vartheta - \frac{\mu_0 I_0 d\, ik}{4\pi r}e^{-ikr}\cos\vartheta - 2\frac{\mu_0 I_0 d e^{-ikr}}{4\pi r^2}\cos\vartheta$$

$$= -\frac{\mu_0 I_0 d}{4\pi}e^{-ikr}\cos\vartheta\left(+\frac{1}{r^2} + \frac{ik}{r}\right)$$

And then the gradient

$$\nabla\cdot(\nabla\cdot\vec{A}) = \frac{\partial(\nabla\cdot\vec{A})}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial(\nabla\cdot\vec{A})}{\partial\vartheta}\hat{\vartheta} \qquad \left(\frac{\partial(\nabla\cdot\vec{A})}{\partial\phi} = 0\right)$$

$$= -\frac{\mu_0 I_0 d}{4\pi}\left[(-ik)e^{-ikr}\cos\vartheta\left(-\frac{1}{r^2} + \frac{ik}{r}\right) + e^{-ikr}\cos\vartheta\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\right]\hat{r} +$$

$$+ \frac{\mu_0 I_0 d}{4\pi}e^{-ikr}\sin\vartheta\left(-\frac{1}{r^3} + \frac{ik}{r^2}\right)\hat{\vartheta}$$

$$= \frac{\mu_0 I_0 d}{4\pi}e^{-ikr}\cos\vartheta\left(-\frac{k^2}{r} + \frac{2ik}{r^2} + \frac{2}{r^3}\right)\hat{r} + \frac{\mu_0 I_0 d}{4\pi}e^{-ikr}\sin\vartheta\left[\frac{1}{r^3} + \frac{ik}{r^2}\right]\hat{\vartheta}$$

Substituting on (8.21) and combining we get the desired equations.

(8.2). Assuming spherical propagation of the waves, we have

$$\int \langle P \rangle \, dV = 10^3 W \quad \text{or} \quad \langle P \rangle = \frac{10^3 W}{4\pi r^2}$$

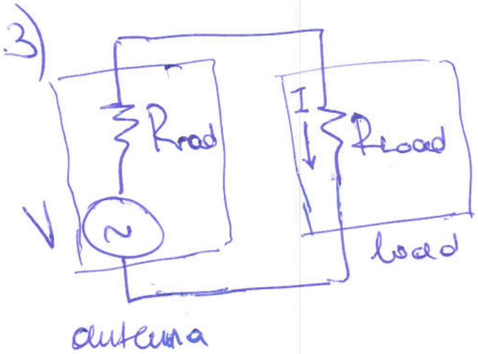
So for $r = 10^3 m \Rightarrow \langle P \rangle = \frac{10^3 W}{4\pi \, 10^6 m^2} = 7.96 \times 10^{-5} \frac{W}{m^2}$

· We know $\vec{P} = \vec{E} \times \vec{H} = \vec{E} \times \left(\sqrt{\frac{\varepsilon_0}{\mu_0}} \hat{k} \times \vec{E} \right)$

So $\langle P \rangle = \sqrt{\frac{\varepsilon_0}{\mu_0}} \langle E \rangle^2 = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_{peak}^2$ (from Chap 6 Problem 5)

from where $E_{peak} = \sqrt{2 \langle P \rangle \sqrt{\frac{\mu_0}{\varepsilon_0}}} = \sqrt{2 \cdot 7.96 \times 10^{-6} \cdot 377} \ V = 0.245 \ V$

(8.3)



The antenna is inducing current equal to $I = \frac{V}{R_{rad} + R_{load}}$

The power reaching the load is

$$W_{Load} = I^2 R_{load} = V^2 \cdot \frac{R_{LOAD}}{(R_{RAD} + R_{LOAD})^2}$$

To find the max we want $\frac{\partial W_{LOAD}}{\partial R_{LOAD}} = 0 \Rightarrow V^2 \left[\frac{1}{(R_{RAD} + R_{LOAD})^2} - \frac{2 R_{LOAD}}{(R_{RAD} + R_{LOAD})^3} \right] = 0$

or $R_{RAD} + R_{LOAD} - 2 R_{LOAD} = 0$

or $\boxed{R_{RAD} = R_{LOAD}}$

(8.4) From (8.48): $G = \max_{\vartheta, \phi} \frac{P(r=1, \vartheta, \phi)}{W / 4\pi}$, from (8.31) $W = \frac{I_0^2 \, n}{3} \sqrt{\frac{\mu_0}{\varepsilon_0}} \left(\frac{d}{\lambda} \right)^2$

and from (8.30) $P = \frac{I_0^2 k^2 d^2}{32 n^2 r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \sin^2 \vartheta \Rightarrow \max_{\vartheta, \phi} P \underset{\vartheta = 1}{=} \frac{I_0^2 d^2 k^2}{32 n^2 r^2} \sqrt{\frac{\mu_0}{\varepsilon_0}}$

So $G = \frac{I_0^2 d^2 k^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \, 3 \cdot 4n}{32 n^2 1^2 \, I_0^2 \, n \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{d^2}{\lambda^2}} = \frac{3k^2}{8 \frac{n^2}{\lambda^2}} = \frac{3 \frac{4n^2}{\lambda^2}}{8 \frac{n^2}{\lambda^2}} = \frac{3}{2}$