3.1 (a) $P_n(x) = {}^nC_x\, p^x (1-p)^{n-x}$

$$P_n(x) = \frac{n!}{(n-x)!\, x!} p^x (1-p)^{n-x}$$

$$\ln [P_n(x)] = \ln(n!) - \ln[(n-x)!] - \ln x! + \ln p^x + \ln[1-p]^{n-x}$$

For large n.

$$\Rightarrow \quad \ln(n!) \approx n \ln n - n \approx n\ln(n)$$

$$\therefore \ln[P_n(x)] = n\ln(n) - (n-x)\ln(n-x) - \ln x!$$
$$+ x\ln p + (n-x)\ln(1-p)$$

$$n\ln(n) \approx n\ln(n-x)$$

$$\therefore \ln[P_n(x)] = x\ln(n-x) - \ln x! + x\ln(p) + (n-x)\ln(1-p)$$

For large n small p $\quad (n-x)\ln(1-p) = -np$

$$\therefore \ln[P_n(x)] = x\ln(n) - \ln x! + x\ln(p) - np.$$

$$P_n(x) = \frac{n^x p^x e^{-np}}{x!} = \frac{e^{-np}(np)^x}{x!} = \frac{e^{-N} N^x}{x!}$$

where $\underline{\underline{N = np}}$

3.1 (b) We have $\langle x(x-1)\cdots(x-m+1)\rangle$

$$= \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{x!} x(x-1)\cdots(x-m+1)$$

For $x \leq m$ ~~sum~~ $x(x-1)\cdots(x-m+1) = 0$.

$$\Rightarrow \sum_{x=m}^{\infty} \frac{e^{-N} N^{x}}{x!} x(x-1)\cdots(x-m+1)$$

$$\Rightarrow \sum_{x=m}^{\infty} \frac{e^{-N} N^{x}}{(x-m)(x-m-1)\cdots 1}$$

Let ~~x=m~~ $y = x-m$, $x = y+m$

$$\Rightarrow \sum_{y=0}^{\infty} \frac{e^{-N} N^{y+m}}{y(y-1)\cdots 1}$$

$$\Rightarrow \sum_{y=0}^{\infty} \frac{e^{-N} N^{y} \cdot N^{m}}{y!} = N^{m} \sum_{y=0}^{\infty} \frac{e^{-N} N^{m}}{y!} = N^{m} e^{-N} e^{N}$$

$$\boxed{= N^{m}}$$

3.1 (c) $\langle x \rangle = \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{x!} x = \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{(x-1)!} = \sum_{x=1}^{\infty} \frac{e^{-N} N^{x-1} N}{(x-1)!}$

$$= N \sum_{x-1=0}^{\infty} \frac{e^{-N} N^{x-1}}{(x-1)!} = N e^{-N} e^{N} = N$$

$$\langle x^2 \rangle = \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{x!} x^2 = \sum_{x=0}^{\infty} \frac{e^{-N} N^{x} x(x-1+1)}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-N} N^{x} x(x-1)}{x!} + \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{x!} x.$$

$$= \sum_{x=0}^{\infty} \frac{e^{-N} N^{x}}{(x-2)!} + N$$

$$= N^2 + N.$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = N^2 + N - N^2 = N$$

$$\sigma = \sqrt{N}$$

$$\frac{\sigma}{\langle x \rangle} = \frac{\sqrt{N}}{N} = 1/\sqrt{N}$$

3.2) Fractional error in estimation of average value is given by Relative Standard deviation

For Poisson distribution $\dfrac{\sigma}{E[x]} = \dfrac{1}{\sqrt{N}}$

Given error $= 1\% = 0.01 = 1/100$

$\sqrt{N} = 100 \Rightarrow N = 10^4$

Also for $\sqrt{N} = 10^{-6} \Rightarrow N = 10^{12}$.

For visible light energy is $E = \dfrac{hc}{\lambda}$

where $\lambda = 700$ nm

For n photons $\dfrac{E}{s} = n\dfrac{hc}{\lambda}$

For $N = 10^4$ $\dfrac{E}{s} = \dfrac{10^4}{s} \times 6.6\times10^{-34}\,J\cdot s \times \dfrac{3\times10^8\,m/s}{7\times10^{-7}\,m}$

$= 2.82\times10^{-30}\times10^{15}\;J/s$

$P = 2.82 \times 10^{-15}\,W$

For $N = 10^{12}$ $P =$ $2.82\times10^{-7}\,W$

3.3 (a) Given $R = 10k\Omega$, $SNR = 20dB$, $\Delta f = 20kHz$

Johnson Noise is given by

$$\langle V_{noise}^2 \rangle = 4kTR\Delta f$$

$$\langle V_{noise}^2 \rangle = 4 \times 1.38 \times 10^{-23}\, J/K \times 300K \times 10000\,\Omega \times 20 \times 10^3\, Hz$$

$$\langle V_{noise}^2 \rangle = 3.3 \times 10^{-12}\, V^2$$

$$V_{noise} = 1.8 \times 10^{-6}\, V$$

$$10 \log\left(\frac{\langle V_{signal}^2 \rangle}{3.3 \times 10^{-12}}\right) = 20$$

$$\log\left(\frac{\langle V_{signal}^2 \rangle}{3.3 \times 10^{-12}}\right) = 2$$

$$\underline{\underline{\langle V_{signal} \rangle = 1.8 \times 10^{-5}\, V}}$$

(b) $$\langle \tfrac{1}{2} C V^2 \rangle = \tfrac{1}{2} kT$$

$$C = \frac{kT}{\langle V^2 \rangle} = \frac{1.38 \times 10^{-3} \times 300}{3.3 \times 10^{-12}}\, F$$

$$= 1.2 \times 10^{-9}\, F = 1.2\, nF$$

(c) $$\langle I_{noise}^2 \rangle = 2q \langle I \rangle \Delta f$$, given $I_{noise, RMS} = 0.01 \langle I \rangle$

$$\therefore 10^{-4} \langle I^2 \rangle = 2q \langle I \rangle \Delta f, \quad \langle I \rangle = \frac{2 \times 1.6 \times 10^{-19} \times 20 \times 10^3}{10^{-4}}\, A$$

$$\langle I \rangle = \cancel{10^{-4} / 2q \times 20kHz} = 64 \times 10^{-12}\, A.$$

3.4 (a) Let probability of being in state 0 at time t be $P_0(t)$ and similarly for state 1 we have $P_1(t)$

So we get $$\frac{dP_0}{dt} = -\alpha P_0 + \beta P_1$$

$$\frac{dP_1}{dt} = \alpha P_0 - \beta P_1$$

$$\frac{d}{dt}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

$$\frac{d}{dt}\vec{P} = A\cdot\vec{P}, \text{ where } A = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$$

(b) Diagonalization

characteristic equation for A is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\alpha-\lambda & \beta \\ \alpha & -\beta-\lambda \end{vmatrix} = 0$$

$$(-\alpha-\lambda)(-\beta-\lambda) - \alpha\beta = 0 \Rightarrow \alpha\beta + \alpha\lambda + \beta\lambda + \lambda^2 - \alpha\beta = 0$$

$$\Rightarrow \lambda^2 + \lambda(\alpha+\beta) = 0 \Rightarrow \lambda(\lambda + \alpha + \beta) = 0$$

Eigen values are $\lambda = 0$ and $\lambda = -(\alpha+\beta)$

For $\lambda = 0$, Eigen vector ~~is given by~~ should satisfy

$$(A - \lambda I)X = 0$$

$$-\alpha x_1 + \beta x_2 = 0$$

$$\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\alpha x_1, -\beta x_2) = 0$$

or $\boxed{x_1 = \frac{+\beta}{\alpha} x_2}$ — (1)

For $\lambda = -(\alpha+\beta)$, Eigen vector is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -\alpha + \alpha + \beta & \beta \\ \alpha & -\beta + \alpha + \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\beta x_1 + \beta x_2 = 0$$

$$\alpha (x_1 + x_2) = 0 \quad \Rightarrow \quad x_1 = -x_2. \quad \text{— (2)}$$

From (1) & (2)

Eigen vectors are $\begin{bmatrix} 1 \\ \alpha/\beta \end{bmatrix}$ and. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So we have $C = \begin{bmatrix} 1 & 1 \\ \alpha/\beta & -1 \end{bmatrix}$

$$\det C = -1 - \alpha/\beta$$

$$C^{-1} = \frac{1}{-(1+\alpha/\beta)} \begin{bmatrix} -1 & -1 \\ -\alpha/\beta & 1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 1/1+\alpha/\beta & 1/1+\alpha/\beta \\ \dfrac{\alpha/\beta}{1+\alpha/\beta} & -1/1+\alpha/\beta \end{bmatrix}$$