4.1 Non-negativity

$H(x) = \Sigma\, p \log(1/p)$

Clearly $p>0$ and $p<1$

$\therefore$ $1/p > 1$ and $\log(1/p) > 0$

$\therefore$ $p \log(1/p) > 0 \Rightarrow \Sigma\, p \log(1/p) \geq 0$

~~Monotonicity~~

Continuity : Since sum of continuous function is continuous let's take $p \log p$.

Let $f = p \log(p)$, where $f = f(p)$

f is continous if for $c - \delta < p < c + \delta$

$f(c) - \delta < f(p) < f(c) + \delta$

$\Rightarrow \quad -\delta < f(p) - f(c) < \delta$.

$|f(p) - f(c)| < \delta \quad$ — (1)

~~Let $p \geq p'$ and~~

~~as~~ we have $c - \delta < p < c + \delta$.

$(c-\delta) \log p < p \log p < (c+\delta) \log p$

clearly $c \log p - \delta < p \log p < c \log p + \delta$. — (2)

$|p \log p| < \delta$

From (1) and (2) $H(x)$ is continuous.

<u>Monotonicity</u> : $H(x) = \sum_{i=1}^{n} p \log(1/p)$

Consider $p \log 1/p$

~~For small σ~~ $\cancel{(p+\sigma) \log(p+\sigma)}$)

For small σ we have $(p+\sigma) \log\left(1/_{p+\sigma}\right)$

If σ is positive $p+\sigma > p \Rightarrow 1/_{p+\sigma} < 1/p$

$\Rightarrow$ But p and $p+\sigma$ are less than 1.

4.2. $I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{P(x)P(y)}$

$$= \sum_x \sum_y p(x,y) \log p(x,y) - \sum_x \sum_y p(x,y) \log p(x)$$
$$- \sum_x \sum_y p(x,y) \log p(y)$$

$$= \sum_x \sum_y p(x,y) \log p(x,y) - \sum_x p(x) \log p(x)$$
$$- \sum_y p(y) \log p(y)$$

$$= -H(x,y) + H(x) + H(y)$$

Also $\quad H(x,y) = H(x|y) + H(y)$

$\therefore I(x,y) = H(x) - H(x|y)$

Similarly $\quad H(x,y) = H(y/x) + H(x)$

$\therefore I(x,y) = H(y) - H(y/x)$

4.3 ⟨a⟩ Probability of error with majority voting = P(at least two erroneous bits)

$$= 3P(\text{two erroneous bits}) + P(3 \text{ erroneous bits})$$

$$= 3\epsilon^2(1-\epsilon) + \epsilon^3$$

$$= \epsilon^2[3 - 3\epsilon + \epsilon]$$

$$= \epsilon^2[3 - 2\epsilon]$$

⟨b⟩ Probability of error $= \epsilon'^2[3 - 2\epsilon']$

where $\epsilon' = \epsilon^2[3 - 2\epsilon]$

$$= [\epsilon^2(3-2\epsilon)]^2[3 - 2\epsilon^2(3-2\epsilon)]$$

$$= \epsilon^4(9 + 4\epsilon^2 - 12\epsilon)(3 - 6\epsilon^2 + 4\epsilon^3)$$

(4) Differential entropy $= -\int_{-\infty}^{\infty} p(x) \log[p(x)]\, dx$

For a gaussian process $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

where μ = mean σ^2 = variance

$$-\int_{-\infty}^{\infty} p(x) \log p(x)\, dx = -\int_{-\infty}^{\infty} p(x) \log\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right] dx$$

$$= -\int_{-\infty}^{\infty} p(x) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) dx - \int_{-\infty}^{\infty} p(x)\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \int_{-\infty}^{\infty} p(x) \log\left(\sqrt{2\pi\sigma^2}\right) dx + \int_{-\infty}^{\infty} p(x) \frac{(x^2+\mu^2-2\mu x)}{2\sigma^2}\, dx$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \int_{-\infty}^{\infty} \frac{p(x)\, x^2\, dx}{2\sigma^2} + \int_{-\infty}^{\infty} \frac{p(x)\, \mu^2\, dx}{2\sigma^2} - \int_{-\infty}^{\infty} \frac{p(x) \cdot 2\mu x\, dx}{2\sigma^2}$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{1}{2\sigma^2}\langle x^2\rangle + \frac{\mu^2}{2\sigma^2} - \frac{2\mu}{2\sigma^2}\langle x\rangle$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{1}{2\sigma^2}\left(\langle x\rangle + \sigma^2\right) + \frac{\mu^2}{2\sigma^2} - \frac{2\mu^2}{2\sigma^2}$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{\mu + \sigma^2 - \mu^2}{2\sigma^2}$$

$$= -\int_{-\infty}^{\infty} P(x) \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) dx + \int_{-\infty}^{\infty} P(x) \frac{(x-\mu)^2}{2\sigma^2} dx$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} P(x) (x-\mu)^2 dx.$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{1}{2\sigma^2} \langle (x-\mu)^2 \rangle$$

$$= \log\left(\sqrt{2\pi\sigma^2}\right) + \frac{1}{2\sigma^2} \sigma^2$$

$$= \log_e\left(\sqrt{2\pi\sigma^2}\right) + 1/2$$

$$= 1/2 \log\left(2\pi\sigma^2\right) + 1/2 \log_e e$$

$$= \underline{1/2 \log\left(2\pi e\sigma^2\right)}$$

4.5 (a) $\Delta F = 3300\ Hz$ $\qquad SNR = 20\ dB = 10\log(S/N)$.

$$\text{Capacity} = \Delta F \log_2\left(1 + \frac{S}{N}\right) \text{ bits/second}$$

$$= 3300 \log_2(1 + 10^2) \text{ bits/second}$$

$$= 21945 \text{ bits/second.}$$

(b). $\text{Capacity} = 1\ Gbit/s = 10^9\ bit/s = 3300 \log_2(1 + S/N)$

$$\log_2(1 + S/N) = \frac{10^9}{3.3 \times 10^3}$$

$$\log_{10}(1 + S/N) = \frac{10^9 \times \log_{10} 2}{3.3 \times 10^3} = \frac{10^9 \times 3 \times 10^{-1}}{3.3 \times 10^3}$$

$$= 10^5$$

$$10\log(S/N) \approx 10^6\ dB$$

$$\underline{\underline{SNR = 10^6\ dB}}$$

4.6) $F(x_1, x_2 \dots x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$

$$\langle F(x_1, x_2 \dots x_n) = \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i \right\rangle = 1/n \sum_{i=1}^{n} \langle x_i \rangle$$

$$= 1/n \sum_{i=1}^{n} x_0 = 1/n \cdot n \cdot x_0 = \underline{\underline{x_0}}$$

Hence the estimator is biased.

Cramér-Rao Inequality states $\sigma^2(F) \geqslant 1/J(\alpha)$

Here $\alpha = x_0$

Variance of estimator $= \sigma^2(F) = \left\langle \left[\frac{1}{n} \sum_{i=1}^{n} (x_i - x_0)\right]^2 \right\rangle$

$$= 1/n^2 \sum_{i=1}^{n} \langle (x_i - x_0)^2 \rangle$$

$$= 1/n^2 \cdot \sum_{i=1}^{n} \sigma^2$$

$$= 1/n^2 \cdot n \cdot \sigma^2 = \underline{\underline{\sigma^2/n}}$$