4.1 $\quad H(x) = \sum_{i}^{X} p(x_i) \log p(x_i)$

Continuous: (epsilon-delta) for $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$|(p+\Delta) - p| < \delta$

$\Rightarrow \quad \Delta < \delta$

with $f(x) = p \log p$,

$\underbrace{|(p+\Delta)\log(p+\Delta) - p\log p|}_{=0 \,<\, \varepsilon} < \varepsilon$

when $\Delta \to 0$, satisfied.

Non Negativity.

$0 < p < 1$ (defn of prop)

$-\sum p \log p \Big|_{0<p<1} = -\sum p \underbrace{(-ve)} > 0$

Monotonicity

$$X' > X \Rightarrow H(X') > H(X)$$

if all states equally likely, prob(1 state) $= \frac{1}{X}$

$$\therefore H(X') = -\sum_{i}^{X'} \frac{1}{X'} \log\left(\frac{1}{X'}\right) = -\log\left(\frac{1}{X'}\right)$$

$$\underbrace{-\underbrace{\log\left(\frac{1}{X'}\right)}_{\text{more -ve}}}_{\text{more +ve}} > -\log\left(\frac{1}{X}\right) \quad \text{if } X' > X$$

Independence.

$$H(x,y) = -\sum_{x\,y} p(x,y) \log p(x,y)$$

$p(x,y) = p(x)p(y)$ when $x \perp y$

$$= -\sum_{x\,y} p(x)p(y) \log p(x)p(y)$$

$$= -\sum_{x\,y} (x)p(y)\left(\log p(x) + \log(p(y))\right)$$

$$= -\sum_{x\,y} \left[p(x)\log p(x)\right]p(y) + -\sum_{x\,y} p(y)\log p(y)\, p(x)$$

$$= \sum_{x} p(x) \log p(x) \underbrace{\sum_{y} p(y)}_{=1} + \ldots$$

$$= H(x) + H(y)$$

4.2 Prove $I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$

y $H(x) + H(y) - H(x,y)$

$$= -\sum_x p(x) \log (x) - \sum_y p(y) \log p(y) + \underbrace{\sum_{xy} p(x,y) \log p(x,y)}_{\ldots ?}$$

$$= -\sum_x p(x,y) \log p(x) - \sum_{xy} p(x,y) \log p(y) + \ldots$$

$$= \sum_{xy} p(x,y) \left\{ \log p(x,y) - \log p(x) - \log p(y) \right\}$$

$$= \ldots \frac{p(x,y)}{p(x)\, p(y)}$$

2/ Same, symetric

3/ $H(x) - H(x|y) = -\underbrace{\sum_x p(x) \log p(x)}_{\to \sum_{xy} p(x,y)} + \sum_{xy} p(x|y) \log p(x|y)$

$$= \ldots + \sum_{xy} p(x,y) \log p(x|y)$$

$$= \sum_{xy} p(x,y) \left(-\log p(x) + \log p(x|y) \right)$$

$$= \sum_{xy} p(x,y) \underbrace{\log \frac{p(x|y)\cdot p(y)}{p(x)\cdot p(x)}}_{= \frac{p(x,y)}{p(x)\,p(y)}}$$

4.3

binary channel, ε prob of making error.

$$x = \begin{cases} 0 \\ 1 \end{cases}$$

$$y = \begin{cases} x & \text{w/ prob } 1-\varepsilon \\ 1-x & \text{w/ prob } \varepsilon \end{cases}$$

$1-x$ is inverse of x as x is binary.

1/ $\text{prob (error)} = (1-x)\varepsilon \cdot (1-x)\varepsilon \cdot x(1-\varepsilon) \times {}^3C_2$ + all error

$$= 3(1-x)^2 x \varepsilon^2 (1-\varepsilon) + (1-x)^3 \varepsilon^3$$

2/ $\text{prob (not error)} = x^3 \varepsilon^3$

1/ prob (error) = all error + $\overset{\text{3 possibilities}}{3}$(2 wrong, 1 right)

$$= \varepsilon^3 + 3\varepsilon^2(1-\varepsilon)$$

$$= 3\varepsilon^2 - 2\varepsilon^3 = O(\varepsilon^2)$$

2/ prob (no error) = all right + 3(2 right, 1 wrong)

$$= (1-\varepsilon)^3 + 3(1-\varepsilon)^2\varepsilon$$

$$= 2x^3 - 3x^2 + 1$$

prob (error in majority voting) = all error + 3(2 error, 1 right)

$$= (3\varepsilon^2 - 2\varepsilon^3)^3 + 3\left\{(3\varepsilon^2-2\varepsilon^3)^2 \times (2x^3 - 3x^2 + 1)\right\}$$

$$= 16\varepsilon^9 - 72\varepsilon^8 + 108\varepsilon^7 - 42\varepsilon^6 + 36\varepsilon^5 + 27\varepsilon^4$$

$$= O(\varepsilon^4)$$

3/ 1 majority voting, error $\propto \approx \varepsilon^2$

1 maj on maj, error $\propto \approx \varepsilon^4$ ie $(\varepsilon^2)^N$

N maj ... on maj, error $\propto \approx (\varepsilon^2)^N = O(\varepsilon^{2N})$

4.4 Diff entropy of Gaussian channel

$$-\int_{-\infty}^{\infty} p(x)\log p(x)\,dx \leftarrow \text{diff. entropy.}$$

$$p(x) = \frac{1}{\underbrace{2\pi\sigma^2}_{\alpha}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \frac{1}{\beta} = \alpha e^{-(x-\mu)^2\beta}$$

$$-\int_{-\infty}^{\infty} p(x)\log p(x)\,dx = -\int^{\infty} \alpha e^{-(x-\mu)^2\beta}\left(\log\alpha + -(x-\mu)^2\beta\right)dx$$

$$= -\log\alpha - \alpha\int^{\infty} -(\underbrace{x-\mu}_{y})^2\beta\, e^{-(x-\mu)^2\beta}\,dx \qquad y = x-\mu \quad dy = dx$$

$$= \dots - \alpha\int^{\infty} -\beta y^2 e^{-y^2\beta}\,dy \quad -\frac{\sqrt{\pi}}{2\sqrt{\beta}} \quad \sqrt{2}\,\sigma$$

$$= -\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \frac{1}{\sqrt{2\pi\sigma^2}}\frac{\sqrt{\pi}\sqrt{2\sigma^2}}{2} \quad \frac{1}{2}$$

$$= \log\sqrt{2\pi\sigma^2} + \frac{1}{2}$$

4.5 a) $C = \underbrace{\Delta f}_{3300} \log_2\left(1 + \underbrace{\frac{S}{N}}_{10^{20/10}}\right)$ only not $\frac{S}{N_0 \Delta f}$?

$N = N_0 \Delta f$?

$= 22000$ bits/s

b) $10^9 = 3300 \log_2\left(1 + \frac{S}{N}\right)$

$$\Rightarrow \frac{S}{N} = 2^\wedge\left(\frac{10^9}{33000}\right) - 1 \approx 2^{3\times10^5}$$

$$\approx 10^{10^5}$$

in dB, $10 \log 10^{10^5}$

$$= 10^6 \text{ dB}$$

4.6

$$f(x) = \frac{1}{n}\sum_i^n x_i$$

~~is unbiased?~~ ✓

achieves Cramer-Rao bound?

unbiased

Unbiased if $\langle f(x)\rangle - x_0 = 0$

$$\langle f(x)\rangle = \sum_i^n \frac{1}{n} \underbrace{\sum \frac{1}{n} x_i}_{\langle x_i\rangle = x_0} = \langle x_0\rangle = x_0$$

Cramer-Rao bound: $\sigma^2(f) \geq \frac{1}{J(x_0)}$

$$\sigma^2(f) = \langle (f - x_0)^2 \rangle$$

$$= \left\langle \left(\left\{ \frac{1}{n} \sum_i x_i \right\} - x_0 \right)^2 \right\rangle$$

$$= \left\langle \left(\frac{1}{n} \sum (x_i - x_0) \right)^2 \right\rangle$$

$$= \frac{1}{n^2} \underbrace{\sum (x_i - x_0)^2}_{n\sigma^2}$$

$$= \frac{\sigma^2}{n}$$

$$J(\alpha) = \left\langle \left[\frac{\partial}{\partial \alpha} \log P_\alpha(x) \right]^2 \right\rangle$$

def^n

$$J_N(x_0) = n\, J(x_0)$$

$$= n \left\langle \left[\frac{\partial}{\partial \alpha} \log P(x_0, \sigma^2) \right]^2 \right\rangle$$

$$= n \left\langle \left[\frac{x - x_0}{\sigma^2} \right]^2 \right\rangle$$

$$= n \left\langle \frac{(x - x_0)^2}{\sigma^4} \right\rangle$$

$$= n \cdot \frac{1}{n} \sum_i^n \frac{(x_i - x_0)^2}{\sigma^4}$$

$$= \frac{1}{\sigma^4} \underbrace{\sum_i^n (x_i - x_0)^2}_{n\sigma^2}$$

$$= \frac{n}{\sigma^2}$$

$$\therefore \quad \sigma^2(f) = \frac{1}{J_N(x_0)}$$