8.1 Retarded potential solution with $\beta=1$ (Anderson, 1992);

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x})\, e^{-ik|\vec{r}-\vec{x}|}\frac{d\vec{x}}{|\vec{r}-\vec{x}|} \quad (1)$$

In our case, we have a thin wire carrying a periodic current with a constant amplitude,

$$\vec{J} = I_0\, \delta(x,y)\, \hat{z} \quad (2)$$

Because the wire has an infinitesimal length d in the $\hat{z}$ direction, the vector potential can be read off from Eq. (1) to be,

$$(3)\quad \vec{A}(r) = \frac{\mu_0}{4\pi}\int I_0\, \delta(x,y)\, \hat{z}\, e^{-ik|\vec{r}-\vec{x}|}\frac{d\vec{x}}{|\vec{r}-\vec{x}|} \Rightarrow$$

?

$$(4)\quad \vec{A}(\vec{r}) = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r}\hat{z},$$ with $r = |\vec{x}-\vec{x}'|$ the distance from the source

Since in spherical coordinates (r, θ, ϕ) the unit normal;

$$\hat{z} = \cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}, \quad (5)$$

the vector potential around the wire is:

$$A_r = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r} \cos\theta \qquad (6)$$

$$A_\theta = -\mu_0 \frac{I_0 d e^{-ikr}}{4\pi r} \sin\theta \qquad (7)$$

Since we know from Eq 8.21 (book) that a periodic electric field can be written in terms of the vector potential alone,

$$\vec{E} = \frac{1}{i\omega\mu_0\epsilon_0} \nabla(\nabla\cdot\vec{A}) - i\omega\vec{A} \qquad (8)$$

we need to compute in spherical coordinates:

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} \Rightarrow$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{\cancel{2}}\mu_0 \frac{I_0 d e^{-ikr}}{4\pi \cancel{r}}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\mu_0 \frac{I_0 d e^{-ikr}}{4\pi r}\sin^2\theta\right) \Rightarrow$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\mu_0 \frac{I_0 d e^{-ikr}}{4\pi}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\mu_0 \frac{I_0 d e^{-ikr}}{4\pi r}\sin^2\theta\right) \Rightarrow$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\mu_0 I_0 d}{4\pi}\cos\theta\, r e^{-ikr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{\mu_0 I_0 d}{4\pi r} e^{-ikr}\sin^2\theta\right) \Rightarrow$$

We set $\lambda = \frac{\mu_0 I_0 d}{4\pi}$ and,

$$= \frac{\lambda}{r^2} e^{-ikr} \cos\vartheta - \lambda ik e^{-ikr} \cos\vartheta - 2\frac{\lambda}{r^2} e^{-ikr} \cos\vartheta$$

$$= \lambda e^{-ikr} \cos\vartheta \left(\frac{1}{r^2} + \frac{ik}{r} \right) \quad (9)$$

and then the gradient of the divergence:

$$\nabla \cdot (\nabla \cdot \vec{A}) = \frac{\partial (\nabla \cdot \vec{A})}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial (\nabla \cdot \vec{A})}{\partial \vartheta} \hat{\vartheta} \Rightarrow$$

$$\vdots$$

$$= \lambda e^{-ikr} \cos\vartheta \left(-\frac{k^2}{r} + \frac{2ik}{r^2} + \frac{2}{r^3} \right) \hat{r} + \lambda e^{-ikr} \sin\vartheta \left[\frac{1}{r^3} + \frac{ik}{r^2} \right] \hat{\vartheta} \quad (10)$$

Substituting (9), (10) in (8) we get the components of the electrical field

$$E_\vartheta = \lambda e^{-ikr} \left(\frac{i\omega\mu_0}{r} + \frac{1}{r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{1}{i\omega\epsilon_0 r^3} \right) \sin\vartheta$$

$$E_r = \lambda e^{-ikr} \left(\frac{2}{r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{2}{i\omega\epsilon_0 r^3} \right) \cos\vartheta$$

8.2. The E- & B-fields of a plane EM wave are $\perp$ to each other & to the direction of travel $\hat{k}$, Their ratio

$$\frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 377\Omega$$ has the units resistance and defines the impedance of free space.

From Problem 5.5 in book we found that for a peak value,

$$\langle |\vec{P}| \rangle = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0^2 \quad (1)$$

We are given that,

$$\int_V \langle P \rangle dV = 10^3 W \Rightarrow \langle P \rangle = \frac{10^3 W}{4\pi r^2}$$ and for $r = 1km$,

we get that: $\langle P \rangle = 7.96 \times 10^{-5} W/m^2 \quad (2)$

Explaining a bit more how we got (1):

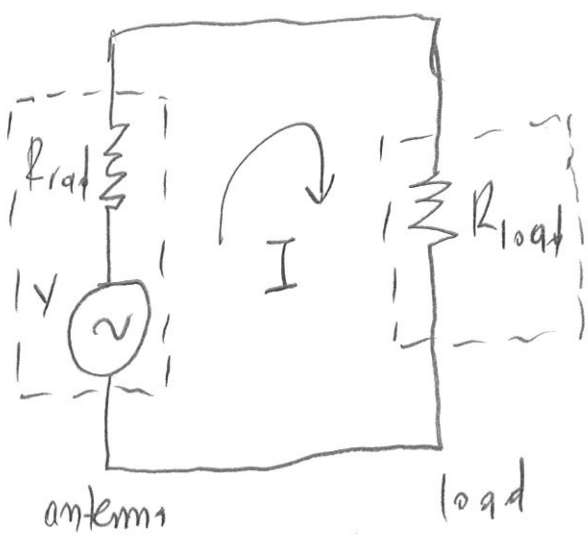
$$\vec{P} = \vec{E} \times \vec{H} = \vec{E} \times \left(\sqrt{\frac{\epsilon_0}{\mu_0}}\, \hat{k} \times \vec{E} \right).$$ Thus,

$\langle P \rangle = \sqrt{\frac{\epsilon_0}{\mu_0}} \langle E \rangle^2$ and the peak value for $t = T$ gives (1):

$$\langle |\vec{P}| \rangle = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_{PEAK}^2$$ from where:

$$E_{PEAK} = \sqrt{2\langle P \rangle \sqrt{\frac{\mu_0}{\epsilon_0}}} = \sqrt{2.796 \times 10^{-6} \cdot 377}\ V = 0.245 V$$

8.3



The sketch shows the effective circuit of an antenna used as a receiver. The radiation induces a voltage V across the antenna terminals, which appears to the load as an ideal generator in series with the antenna's radiation resistance.

Antenna-induced current:
$$I = \frac{V}{R_{rad} + R_{LOAD}} \quad (1)$$

Power at the load:
$$W_{LOAD} = I^2 R_{LOAD} \overset{(1)}{\Longrightarrow}$$

$$W_{LOAD} = V^2 \cdot \frac{R_{LOAD}}{(R_{RAD} + R_{LOAD})^2}$$

To find maximum power we need to solve for:

$$\frac{\partial W_{LOAD}}{\partial R_{LOAD}} = 0 \Rightarrow \frac{\partial \left(V^2 \frac{R_{LOAD}}{R_{RAD} + R_{LOAD}} \right)}{\partial R_{LOAD}} = 0 \Longrightarrow \ldots \boxed{R_{RAD} = R_{LOAD}}$$

9.4 Antenna gain (given by Eq. 7.48 book):

$$G \equiv \max_{\theta,\phi} \frac{P(r=1,\theta,\phi)}{W/4\pi}, \quad (1)$$

where: . W → total energy radiated given by Eq. 7.31;

$$W = \frac{I_0^2\pi}{3}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{d}{\lambda}\right)^2 \quad (2)$$

and

. P → poynting vector given by Eq. 7.30;

$$\langle\vec{P}\rangle = \hat{r}\frac{I_0^2 k^2 d^2}{32\pi^2 r^2}\sqrt{\frac{\mu_0}{\epsilon_0}}\sin^2\theta \quad (3)$$

Thus,

$$(1) \xRightarrow[(3)]{(2)} \quad G = \dots =$$