(6.2.) (a) Divergence theorem (or Gauss's Theorem)

$$\int_V \nabla\cdot\vec{E}\,dV = \int_S \vec{E}\cdot d\vec{A} \quad (1)$$

States that: "the integral of a derivative (in this case a divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface S that bounds the volume"

Geometrical interpretation: "If we had $\vec{v}$ representing the flow of an incompressible fluid, then the FLUX of $\vec{v}$ (right side of Eq 1) is the total amount of fluid passing out through the surface, per unit time. Now the divergence measures the "spreading out" of the vectors from a point (faucet-pouring out liquid)

Thus, $\int$ (faucets within the volume) = $\oint$ (flow out through the surface)

Going back to electrostatics,

the quantitative statement of Gauss's law is for any closed surface then we have,

$$\oint E\cdot dA = \frac{1}{\epsilon_0} Q_{enclosed} \quad (2)$$

As it stands, Gauss's law is an integral equation, but we can easily turn it into a differential one, by applying the divergence theorem;

$$\oint_S \vec{E} \cdot dA = \int_V (\nabla \cdot \vec{E})\, dV \quad (3)$$

Rewriting Q_{enc} in terms of charge density ρ, we have;

$$Q_{enc} = \int_V \rho \, dV \quad (4)$$

From (2) + (3), using (4);

$$\int_S E \cdot dA = \int_V (\nabla \cdot \vec{E})\, dV = \int_V \left(\frac{\rho}{\epsilon_0}\right) dV$$

And since this holds for any volume, the integrands must be equal;

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

Similarly, lets apply the divergence theorem in reverse:

$$\int_V \nabla \cdot \vec{E}\, dV = \int_S \vec{E} \cdot dA = \frac{1}{\epsilon_0} \int \rho \, dV = \frac{1}{\epsilon_0} Q_{enc}.$$

Step 1: We need to find the electric field of an infinite plane that carries a uniform charge σ:

Using Gauss's law to this surface:

$$\int E \cdot dA = \frac{1}{\epsilon_0} Q_{enc}$$

In this case $Q_{enc} = \sigma A$, where A is the area of the lid of the plate.

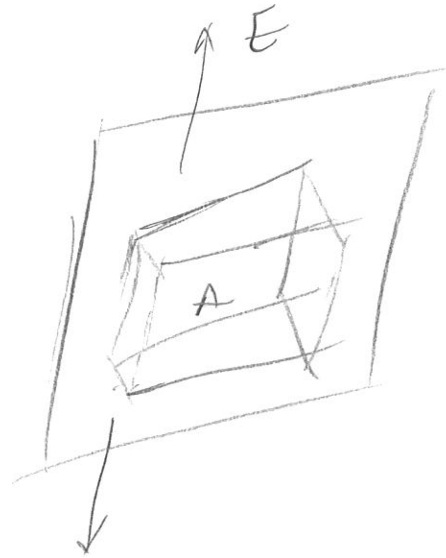
By symmetry, $\vec{E}$ points away from the plane (upward for points above, downward for points below). So the top and bottom surfaces yield,

$$\int E \cdot dA = 2A|E|,$$ whereas the sides contribute nothing.

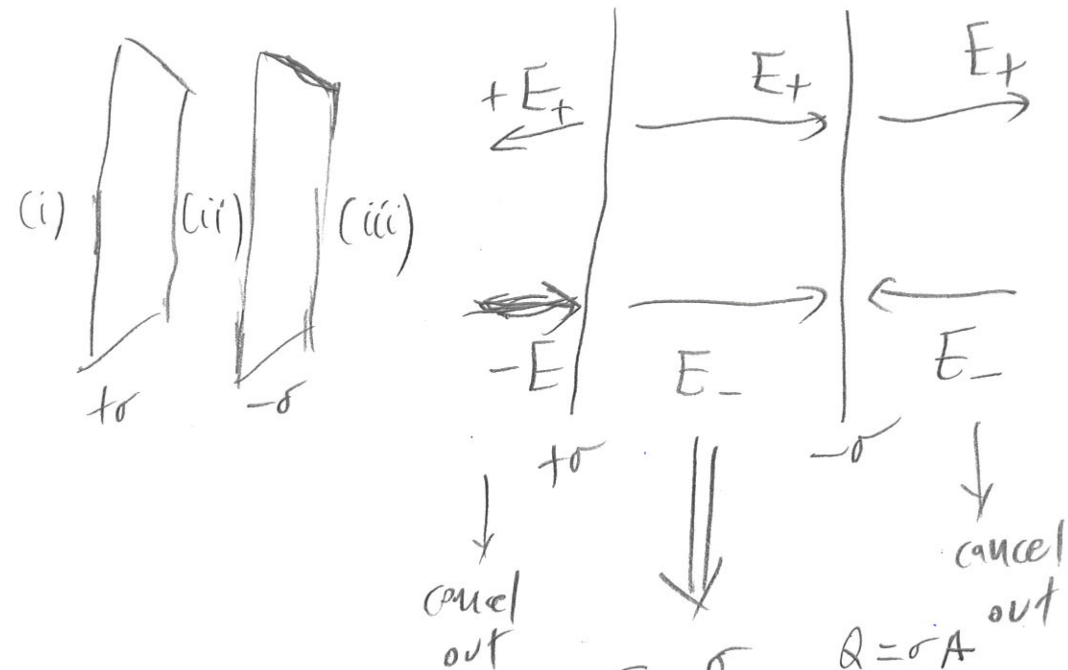
Thus, $2A|E| = \frac{1}{\epsilon_0} \sigma A \Rightarrow$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$, where

$\hat{n}$ is a unit vector pointing away from a surface



Step 2: Now lets do the same for 2 infinite parallel planes carrying equal but opposite uniform charges densities $\pm\sigma$:



$$E = \frac{\sigma}{\epsilon_0} \Rightarrow \qquad Q = \sigma A$$

$$\boxed{E = \frac{Q}{\epsilon_0 A}}$$

Also, $\nabla \times \vec{E} = 0$, thus:

$$\vec{E} = -\nabla V \Rightarrow$$

$$-\int \vec{E} \cdot dl = V \Rightarrow$$

$$V = -\frac{Q}{\epsilon_0 A} \int d\vec{l} \Rightarrow$$

$$\boxed{Q = \frac{\epsilon A V}{d}} \quad \text{and} \quad \boxed{C = \frac{Q}{V} = \frac{\epsilon A}{d}}$$

(b) $$\int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} = \int \epsilon \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} \Rightarrow$$

$$= \int \epsilon \frac{\partial}{\partial t}\left(\frac{V}{d}\right) \cdot A$$

$$= \frac{\epsilon A}{d} \frac{\partial V}{\partial t}$$

$$\frac{\partial \vec{D}}{\partial t} = C \frac{\partial V}{\partial t} \text{ (displacement current)}$$

(c) Energy density of capacitor $\approx \vec{D} \cdot \vec{E}$

Stored energy $$U = \frac{1}{2} \int_V \vec{D} \cdot \vec{E} \, dV$$

$$= \frac{1}{2} \int_V \epsilon E E \, dV$$

$$= \frac{1}{2} \epsilon \int_V \frac{Q^2}{\epsilon^2 A^2} dV$$

$$= \frac{1}{2} \frac{Q^2}{\epsilon A^2} \int_V dV$$

$$= \frac{1}{2} \frac{Q^2}{\epsilon A^{\cancel{2}}} \cancel{A} d =$$

$$= \frac{1}{2} \frac{\epsilon A}{d} \cdot \frac{Q^2}{\epsilon^2 A^2} d^2 = \frac{1}{2} C (E^2 d^2) = \frac{1}{2} C V^2$$

OR

How much work does it take to charge a capacitor up to a final amount Q? Suppose that at some intermediate stage in the process the charge on the + side is q, so that the potential difference is q/C. According to ~~Eq~~:

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q[V(b) - V(a)] \Rightarrow$$

$$V(b) - V(a) = \frac{W}{Q}$$

the work you must do to transport the next piece of charge dq, is

$$dW = \left(\frac{q}{C}\right) dq$$

The total work necessary, then to go from $q=0$ to $q=Q$ is:

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} \xRightarrow{Q=CV}$$

$$W = \frac{1}{2} CV^2,$$ where V is the final potential of the capacitor

(d)

$$\left.\begin{array}{l} C = \epsilon \dfrac{A}{d} \Rightarrow A = \dfrac{Cd}{\epsilon} \quad (1) \\[2ex] E = \dfrac{1}{2} C V^2 \Rightarrow C = \dfrac{2E}{V^2} \quad (2) \end{array}\right\} \Rightarrow A = \left(\dfrac{2E}{V^2}\right)\left(\dfrac{d}{\epsilon}\right)$$

(6.3) (a) Stoke's theorem:

$$\int_S \nabla\times\vec{E}\cdot d\vec{A} = \oint_l \vec{E}\cdot d\vec{l}$$

Because $\nabla\cdot\vec{B}=0$, the magnetic field can always be expressed as the curl of a magnetic vector potential $\vec{A}$ ("div curl = 0");

$$\vec{B} = \nabla\times\vec{A} \qquad \nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$$

Using Stoke's theorem over a closed loop gives an integral relationship:

$$\oint_L \vec{A}\cdot d\vec{l} = \int_A B\cdot ds = \Phi_B$$

the magnetic flux through a surface is the integral of the magnetic vector potential around the loop enclosing the surface.

Direction of a contribution $d\vec{A}$ is parallel to a current element $I\,dl$

(6.3) (a). Applying Stoke's theorem to the integral of the magnetic field around a closed loop:

$$\oint_L \vec{B}\cdot d\vec{l} = \int_A \nabla\times\vec{B}\cdot d\vec{s} = \mu_0 \int_A \vec{J}\cdot d\vec{A}$$

$$\nabla\times\vec{B} = \mu_0 \vec{J}$$

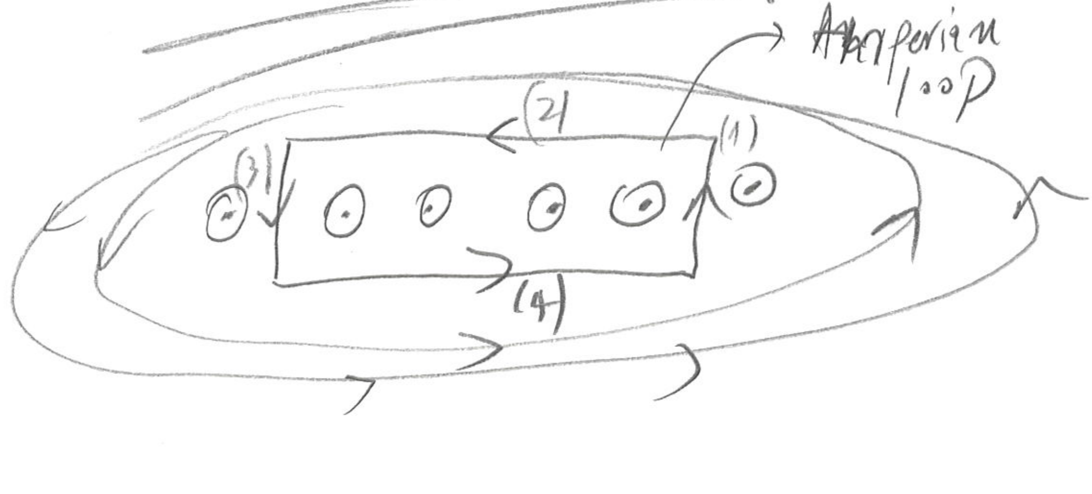
(at any point in space the curl of the magnetic field is proportional to the local current density)

$$\oint \vec{B}\cdot d\vec{l} = I_{enc}\mu_0 \xrightarrow{I_{enc} = I_{solenoid}\cdot N}$$

$$\underset{①}{\cancel{B_{\parallel} h}}^{\,0} + \underset{②}{\cancel{B_{\parallel} L}}^{\,0} + \underset{③}{\cancel{B_{\parallel} h}}^{\,0} + \underset{④}{B_{\parallel} L} = I N \mu_0 \Rightarrow$$

$$B_{inside} L = I N \mu_0 \Rightarrow$$

$$B_{solenoid} = I \left(\frac{N}{L}\right) \mu_0$$



(b) Energy stored; $U = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\,dV \Rightarrow$

$$U = \frac{1}{2}\int_V \mu\vec{H}\cdot\vec{H}\,dV \Rightarrow$$

$$U = \frac{\mu}{2}\int_V H^2\,dV \Rightarrow$$

$$\cancel{U = \frac{\mu}{2}\int n^2 I^2\,dV}$$

$$\boxed{U = \frac{1}{2}\mu n I \pi r^2 l}$$

(c) $F_m = \dfrac{\partial U_m}{\partial r} = \dfrac{\mu n^2 I^2 \pi\, 2rl}{2}$

(6.4). $F = \int \phi \vec{\ell} \times \vec{B}$

$F = q v \times B \Rightarrow dF = \vec{I} d\vec{l} \times \vec{B}$

magnetic field produced by a wire:

$= \vec{I} d\vec{l} \times \mu \vec{H}$

$= \vec{I} dl \, \mu \frac{I}{2\pi r}$ $\quad B = \mu \frac{I}{2\pi r}$

$= \frac{\mu I^2}{2\pi r} dl \Rightarrow$

$F/dl = \frac{\mu I^2}{2\pi r}$

(6.6) $W = -\int_S \vec{P}\cdot d\vec{A} - \int_V \frac{\partial v}{\partial t} dV$

1. energy flux transported across the boundary of the integration volume by the field

change in the energy stored in the volume by the field.

$P \to$ Poynting vector

$v \to$ energy density

" $\int_A P \to$ energy being carried by an EM wave "

" $\int_V v \to$ energy stored in a static field "

$\vec{E} = E_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$c = \omega/|k| = (\mu_0 \epsilon_0)^{-1/2}$

$k = 2\pi/\lambda$

$\cancel{\langle P \rangle}$ $\langle |\vec{P}| \rangle = \langle |\vec{E} \times \vec{H}| \rangle = \langle |\sqrt{\frac{\epsilon_0}{\mu}} E^2 \rangle =$

$= \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 \langle |e^{2i(\vec{k}\vec{x} - \vec{\omega} t)}| \rangle$

But for $t = T$,

- $|e^{i2kx}| = 1$ and
- $|e^{-2i\omega t}| = \langle |\cos 2\omega T - i \sin 2\omega T| \rangle = \langle |\cos 4\pi - i \sin 4\pi| \rangle = 1/2$

thus, $\langle |\vec{P}| \rangle = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 \Rightarrow E_0 \simeq 868 X$