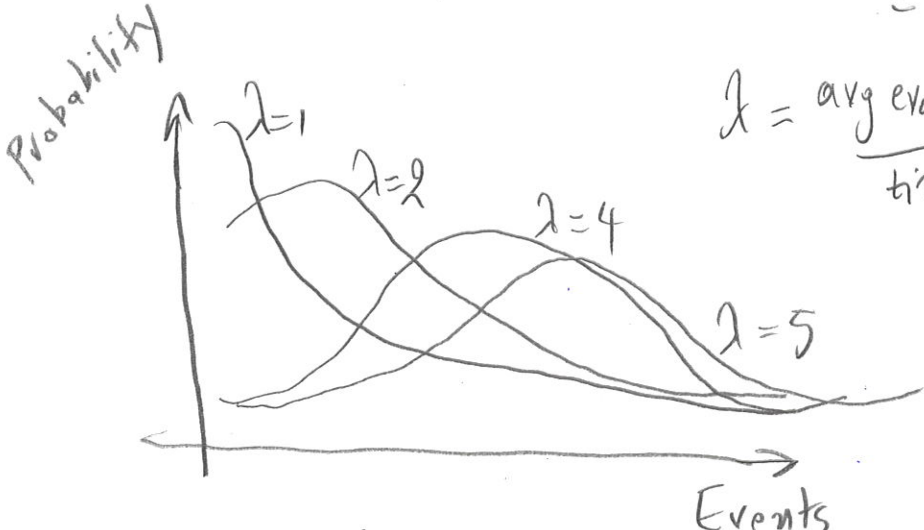
(3,1) (a) The Poisson distribution gives the probability of observing X events in a time period given the length of the period & the avg events per time;

$$P(X) = e^{-\lambda}\frac{\lambda^x}{x!}, \text{ where: } \lambda = N = n\cdot p = (\text{very small intervals})\cdot(\text{prob. of succes}) =$$

= average number of events

$\lambda = \frac{\text{avg events}}{\text{time}} \cdot$ length of time period = rate parameter



Let's say we don't have info about the number of trials n or the probability of success p that led to that rate. Instead we know the AVG NUMBER OF SUCCESSES PER TIME PERIOD OR RATE OF SUCCESSES PER DAY:

$$\lambda = np \Rightarrow p = \frac{\lambda}{n} \quad (2)$$

Binomial distribution: $P_n(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad (1)$ $\quad (1) \xrightarrow{(2)}$

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \frac{n!}{x!(n-x)!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \Rightarrow$$

$$\left(\frac{\lambda^x}{x!}\right)\lim_{n\to\infty} \underbrace{\frac{n!}{x!(n-x)!}\left(\frac{1}{n^x}\right)}_{A} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{B} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-x}}_{C} \quad (3)$$

$$A = \lim_{n\to\infty} \frac{n!}{(n-x)!}\left(\frac{1}{n^x}\right) = \lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-x)(n-x-1)\dots(1)}{(n-x)(n-x-1)\dots(1)}\left(\frac{1}{n^x}\right) =$$

$$= \lim_{n\to\infty} \frac{n(n-1)(n-2)\dots(n-x+1)}{n^x} =$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n}\right)}_{\approx 1}\underbrace{\left(\frac{n-1}{n}\right)}_{\approx 1}\underbrace{\left(\frac{n-2}{n}\right)}_{\approx 1}\dots\underbrace{\left(\frac{n-x+1}{n}\right)}_{\approx 1} \approx 1$$

$$B = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)$$

↳ (Recall that the definition of $e = 2.718\dots$ is given by this expression)

Define this: $x = -\frac{n}{\lambda} \Longrightarrow$

$$B = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-n} = \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{x(-\lambda)} = e^{-\lambda}$$

$$C = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1^{(-x)} \approx 1$$

Substituting A, B, C into (3):

$$\left(\frac{\lambda^x}{x!}\right) \lim_{n\to\infty} \frac{n!}{(n-x)!}\left(\frac{1}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} =$$

$$= \left(\frac{\lambda^x}{x!}\right)(1)\left(e^{-\lambda}\right)(1) \Rightarrow$$

$$\boxed{p(x) = \frac{e^{-\lambda}\lambda^x}{x!}}$$

So, the Poisson distribution is just a special case of the binomial, in which the number of n trials grows to infinity and the chance of success in any particular trial approaches zero.

(b) $\langle x(x-1)\dots(x-m+1)\rangle =$

$$= \sum_{x=m}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} x(x-1)\dots(x-m+1) =$$

$$= e^{-\lambda} \sum_{x=m}^{\infty} \frac{\lambda^x}{x!} x(x-1)\dots(x-m+1) \Rightarrow \text{let } y = x-m$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+m}}{(y+m)!}(y+m)(y+m-1)\dots(y+1) =$$

$$= \lambda^m \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \xrightarrow{\text{Poisson normalized}} \lambda^m$$

c) Using that:

$$\langle x(x-1)\dots(x-m+1)\rangle = \lambda^m$$

and for $m=1$: $\langle x\rangle = \lambda$

$m=2$: $\langle x(x-1)\rangle = \lambda^2 \Rightarrow$

$\langle x^2 - x\rangle = \lambda^2 \Rightarrow$

$\langle x^2\rangle = \lambda^2 + \langle x\rangle$ (1)

We know that the variance for a random variable x:

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 \overset{(1)}{\Rightarrow}$$

$$\sigma^2 = \cancel{\lambda^2} + \langle x\rangle - \cancel{\lambda^2} \Rightarrow$$

$$\sigma^2 = \langle x\rangle \Rightarrow$$

$$\sigma^2 = N \Rightarrow$$

$$\sigma = \sqrt{N}$$

Thus, $\dfrac{\sigma}{\langle x\rangle} = \dfrac{\sqrt{N}}{N} = \dfrac{1}{\sqrt{N}}$

(3.2) We can model the photon arrival at the photodetector with a Poisson distribution:

$$p(x) = \frac{e^{-N} N^x}{x!}$$, where N is the average arrival rate of photons

and x is the random variable representing photon arrival).

Using that $\frac{\sigma}{\langle x \rangle} = \frac{1}{\sqrt{N}}$, we can make a quick estimate of the expected error in a counting measurement. In our case, we know that $\frac{\sigma}{\langle x \rangle}$ should be around 1%. Thus,

$$0.01 = \frac{1}{\sqrt{N}} \Rightarrow N = 10^4 \text{ photons/sec}$$

For 1 ppm = 0.0001% :

$$10^{-4} \cdot 10^{-2} = \frac{1}{\sqrt{N}} \Rightarrow N = 10^{12} \text{ photons/sec}$$

Lets say the photon is blue, $\lambda = 450$ nm. Then,

$$E = h \cdot f = h \cdot \frac{c}{\lambda} = \left(6.62 \cdot 10^{-34} \cdot 3 \cdot 10^{8} \cdot \frac{10^{9}}{450}\right)\left(J \cdot \cancel{s} \cdot \frac{\cancel{m}}{\cancel{s}} / \cancel{m}\right) \Rightarrow$$

$$E = 0.044 \cdot 10^{-17}\ J/\text{photon} \Rightarrow$$

$$E = 4.4 \cdot 10^{-19}\ J/\text{photon}$$

- For $N = 10^{4}$ photons/sec, $E = (4.4 \cdot 10^{-19} \cdot 10^{4})\left(\frac{J}{\cancel{\text{photon}}} \cdot \frac{\cancel{\text{photons}}}{\text{sec}}\right) \Rightarrow$

$$\boxed{E = 4.4 \cdot 10^{-15}\ W}$$

- for $N = 10^{12}$ photons/sec, $\boxed{E = 4.4 \cdot 10^{-7}\ W}$

(3.3) (a) Given: $\Delta f = 20\ kHz$

$R = 10\ k\Omega$

$T = 21°C$ or $T = (273.15 + 21)\ K = 294.15\ K$

We can use the equation for Johnson (or Nyquist) noise,

$$\langle V^2_{noise} \rangle = 4kTR\Delta f, \text{ where } k \rightarrow \text{Boltzmann's constant}$$

$$\Rightarrow \langle V^2_{noise} \rangle = 4 \cdot 1.38 \times 10^{-23} \cdot 294.15 \cdot 10^{4} \cdot 2 \cdot 10^{4} \left(J/\cancel{K} \cdot \cancel{K} \cdot \Omega \cdot s^{-1}\right)$$

$$\Rightarrow \langle V^2_{noise} \rangle = 3{,}247.416 \times 10^{-15}\ V^2 \approx 3.25 \times 10^{-15}\ V^2$$

Any device that detects a signal must content with different types of noise; Johnson noise leads to the generation of voltage noise by an amplifier. Since the power spectral density is flat, the mean square noise mag. ~ to the bandwidth, or the RMS mag. ↑ as $\sqrt{\Delta f}$. Given the practical significance of detecting signals at (and beyond) these limits, it can be more relevant to relate the noise an amplifier introduces to the noise that is input to it. Signals and noise are usually compared on a logarithmic scale to cover a large dynamic range;

the; $$SNR = 20 \log_{10}\left(\frac{V_{RMS,IN}}{V_{RMS,NOISE}}\right) \Rightarrow$$

$$SNR = 20 \log_{10}\left(\frac{V_{RMS,IN}}{\sqrt{\langle V^2_{NOISE}\rangle}}\right) \quad SNR = 20dB \Rightarrow$$

$$\log_{10}\left(\frac{V_{RMS,IN}}{\sqrt{\langle V^2_{NOISE}\rangle}}\right) = 1 \Rightarrow$$

$$V_{RMS,IN} = 10\sqrt{\langle V^2_{NOISE}\rangle} \Rightarrow$$

$$\Rightarrow V_{RMS,IN} = \left(10 \cdot 3.25 \times 10^{-15}\right)^{1/2} V \Rightarrow$$

$$V_{RMS,N} = \left(3.25 \times 10^{-14}\right)^{1/2} V \Rightarrow$$

$$\boxed{V_{RMS,IN} = 1.8 \times 10^{-7} V}$$

(b) Using the equipartition theorem, we can write for the energy in a capacitor that:

$$E_0 = \frac{1}{2} C \langle V_{noise}^2 \rangle \Rightarrow$$

$$\cancel{\frac{1}{2}} kT = \cancel{\frac{1}{2}} C \langle V_{noise}^2 \rangle \Rightarrow$$

$$C = \frac{kT}{\langle V_{noise}^2 \rangle} \Rightarrow$$

$$C = \frac{\left(1.38 \times 10^{-23} \cdot 294.15\right)}{3.25 \times 10^{-15}} \frac{[J/\cancel{K} \cdot \cancel{K}]}{[V^2]} \Rightarrow$$

$$C = 124.9 \times 10^{-8} \left[J/V^2\right] \Rightarrow$$

$$\boxed{C = 1249 \text{ nF}}$$

9) For shot noise we know that,

$$\langle I_{noise}^2 \rangle = 2q \langle I \rangle \Delta f \Rightarrow$$

$$\sqrt{\langle I_{noise}^2 \rangle} = \sqrt{2q \langle I \rangle \Delta f} \Rightarrow$$

$$I_{RMS,noise} = \sqrt{2q \Delta F} \langle I \rangle^{1/2} \Rightarrow$$

$$\langle I \rangle^{1/2} = \frac{I_{RMS,noise}}{\sqrt{2q\Delta F}} \Rightarrow \quad I_{RMS,nois} = 0.01 \langle I \rangle$$

$$\langle I \rangle^{1/2} = \frac{0.01 \langle I \rangle}{\sqrt{2q\Delta f}} \Rightarrow$$

$$0.01 \langle I \rangle = \langle I \rangle^{1/2} \sqrt{2q\Delta F} \Rightarrow$$

$$\langle I \rangle^{1/2} = \frac{\sqrt{2q\Delta F}}{0.01} \Rightarrow$$

$$\langle I \rangle^{1/2} = \frac{\sqrt{2 \cdot 1.6 \times 10^{-19} \cdot 2 \times 10^{4}}}{0.01} \Rightarrow$$

$$\langle I \rangle^{1/2} = \frac{8.19 \times 10^{-7}}{10^{-2}} \Rightarrow$$

$$\langle I \rangle^{1/2} = 8.19 \times 10^{-5} \Rightarrow$$

$$\langle I \rangle = 67.07 \times 10^{-10} \text{ A}$$