

① a) undamped when $\delta = 0$

& presumably when there is no forcing function (homogeneous equation)

b) $m\ddot{x} + \delta\dot{x} + kx = 0$ (homogeneous eq.)

ansatz: $x = A e^{\lambda t}$

characteristic eq: $mr^2 + \delta r + k = 0$

$$r = \frac{-\delta \pm \sqrt{\delta^2 - 4mk}}{2m}$$

$$x_g = \sum_{n=1}^2 A_n e^{r_n t}$$

if $\gamma^2 - 4mk = 0$

$$\gamma = 2\sqrt{mk}$$

system is critically damped

$$\gamma^2 - 4mk < 0$$

$$\gamma < 2\sqrt{mk}$$
 • system is underdamped

- imaginary component leads to oscillatory response

$$\gamma^2 - 4mk > 0$$

$$\gamma > 2\sqrt{mk}$$
 • overdamped.

- no oscillation.

Rewrite: $\ddot{x} + \mu\dot{x} + \omega_n^2 x = 0$

where $\mu = \frac{\gamma}{m}$ $\omega_n^2 = \frac{k}{m}$

$$\tau = \frac{-\mu \pm \sqrt{\mu^2 - 4\omega_n^2}}{2}$$

Rewrite again $\ddot{x} + 2\mu \dot{x} = \frac{\gamma}{m}$

$$\tau = -\mu \pm \sqrt{\mu^2 - \omega_n^2}$$

critical damping : $\mu = \omega_n$

$$\frac{\gamma}{2m} = \sqrt{\frac{k}{m}}$$

$$\gamma = 2\sqrt{km}$$

$$x_g = A_1 e^{(-\mu + \sqrt{\mu^2 - \omega_n^2})t} + A_2 e^{(-\mu - \sqrt{\mu^2 - \omega_n^2})t}$$

$$\text{where } \mu = \frac{\gamma}{2m}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

(c) ansatz : $x_p = C e^{i\omega t}$

substitute into diff. eq : ~~$\ddot{x} + 2\mu \dot{x} + \omega_n^2 x = e^{i\omega t}$~~ $\ddot{x} + 2\mu \dot{x} + \omega_n^2 x = e^{i\omega t}$

$$C e^{i\omega t} (i^2 \omega^2 + 2\mu i\omega + \omega_n^2) = \frac{e^{i\omega t}}{m}$$

$$C (-\omega^2 + \omega_n^2 + 2\mu \omega i) = 1/m$$

$$C = \frac{1}{m(\omega_n^2 - \omega^2) + 2\mu \omega i}$$

$$x_p = \frac{1}{\omega(\omega_n^2 - \omega^2) + 2\mu\omega i} e^{i\omega t}$$

$$\omega = k = \pi, \gamma = 0.1$$

$$\rightarrow \mu = 0.05 \quad \omega_n^2 = 1$$

$$2\mu = 0.1$$

$$x_p = \frac{1}{(1 - \omega^2) + 0.1\omega i} e^{i\omega t}$$

Plot of amplitude \propto phase.

This can be expressed as:

$$x_p = \frac{(1 - \omega^2) - 0.1\omega i}{(1 - \omega^2)^2 + 0.01\omega^2} e^{i\omega t}$$

magnitude

$$|x_p| = \frac{1}{\sqrt{(1 - \omega^2)^2 + 0.01\omega^2}} = \frac{1}{\sqrt{1 - 1.99\omega^2 + \omega^4}}$$

$$\varphi(x_p) = \tan^{-1} \left(\frac{-0.1\omega}{1 - \omega^2} \right)$$

$$(d) \langle E \rangle = \frac{1}{2} m \langle \dot{x} \rangle^2 + \frac{1}{2} k \langle x \rangle^2$$

Assume transients have died out $\Rightarrow x_0 \rightarrow 0$

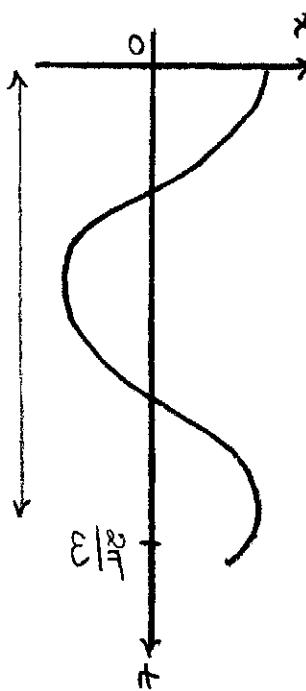
$$\therefore x = C e^{i\omega t} = C \{ \cos \omega t + i \sin \omega t \} \text{ as before}$$

Working with real part only,

$$x = C \cos \omega t \quad \& \quad \dot{x} = -C \omega \sin \omega t$$

$$\langle E \rangle = \frac{1}{2} m C^2 \omega^2 \langle \sin^2 \omega t \rangle + \frac{1}{2} k C^2 \langle \cos^2 \omega t \rangle$$

$\langle \cdot \rangle$ means average over an entire oscillation.



1 cycle except out = 2π

$$\text{time for one cycle} = \frac{\text{angle}}{\text{frequency}} = \frac{2\pi}{\omega}$$

$$\langle \cos^2 \omega t \rangle = \frac{\int_0^{2\pi/\omega} \cos^2 \omega t dt}{2\pi/\omega} = \frac{\frac{1}{2} \left(x + \frac{\sin 2\omega x}{2} \right) \Big|_0^{2\pi/\omega}}{2\pi/\omega} = \frac{\pi/\omega}{2\pi/\omega}$$

$$= \frac{1}{2}.$$

$$\text{similarly, } \langle \sin^2 \omega t \rangle = \frac{1}{2}.$$

$$\langle E \rangle = \frac{1}{4} C^2 (m\omega^2 + \kappa)$$

average over a cycle.

$$\langle E \rangle (\omega) = \frac{1}{4} C^2 (m\omega^2 + \kappa)$$

$$\text{From c) : } C = \frac{1}{m(\omega_n^2 - \omega^2) + 2\mu\omega^2}$$

$$\langle E \rangle = \frac{1}{4} \frac{m\omega^2 + \kappa}{(m\omega_n^2 - m\omega^2 + 2\mu\omega^2)(m\omega_n^2 - m\omega^2 + 2\mu\omega^2)}$$

couple have solved (c) using m, γ, κ .

$$-m\omega^2 C + i\omega\gamma C + \kappa C = 1$$

$$C = \frac{1}{\kappa - m\omega^2 + i\omega\gamma}$$

$$\text{then } \langle E \rangle = \frac{1}{4} \frac{m\omega^2 + \kappa}{(\kappa - m\omega^2 + i\omega\gamma)(\kappa - m\omega^2 + i\omega\gamma)}$$

$$\langle E \rangle = \frac{1}{4} \frac{m\omega^2 + \kappa}{(\kappa - m\omega^2 + i\omega\gamma)^2}$$

- when does this fail to have its peak?

- when does it peak?

we can assume damping is small

$$\gamma \ll \kappa, \omega$$

$$\gamma \rightarrow 0 ?$$

$$\text{if } \gamma \rightarrow 0 \quad \langle \epsilon \rangle = \frac{1}{4} \frac{k + m\omega^2}{k - m\omega^2}$$

if γ is small, $\omega \approx \omega_n$

$$\sqrt{\frac{k}{m}}$$

try to express everything in terms of ω, ω_n

$$\langle \epsilon \rangle = \frac{1}{4} \frac{\omega^2 + \frac{k}{m}}{\left(\frac{k}{m} - \frac{\omega^2}{m} + \frac{i\omega\gamma}{m} \right)^2} = \frac{1}{4} \frac{\omega^2 + \omega_n^2}{\left(\frac{\omega_n^2}{m} - \frac{\omega^2}{m} + \frac{i\omega\gamma}{m} \right)^2}$$

$$= \frac{1}{4} \frac{\omega^2 + \omega_n^2}{\frac{1}{m^2} (\omega_n^2 - \omega^2 + i\omega\gamma)^2}$$

assume $\omega = \omega_n$??

$$\langle \epsilon \rangle = \frac{\frac{1}{4}\omega^2}{\frac{2\omega^2}{m^2} - \frac{i^2\omega^2\gamma^2}{m^2}} = -\frac{1}{4m^2} \frac{2}{\gamma^2}$$

$$\langle \epsilon \rangle = -\frac{1}{2\omega^2\gamma^2}$$

But this is constant. Something's wrong.

coupled mode oscillator.

$$x = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

$$T_{1,2} = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}$$

$$\dot{x} = A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t}$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$E = \frac{1}{2} m (A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t})^2 + \frac{1}{2} k (A_1 e^{r_1 t} + A_2 e^{r_2 t})^2$$

$$\text{period} = \frac{2\pi}{\omega} \quad \text{true for one radian} = \frac{1}{\omega}$$

$$\frac{dE}{dt} = m (A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t}) (A_1 r_1^2 e^{r_1 t}, A_2 r_2^2 e^{r_2 t}) + k (A_1 e^{r_1 t} + A_2 e^{r_2 t}) (A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t})$$

$$= (A_1 r_1 e^{r_1 t} + A_2 r_2 e^{r_2 t}) [m A_1 r_1^2 e^{r_1 t} + m A_2 r_2^2 e^{r_2 t} \dots]$$

$$(e) \quad m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$$

$$m(s^2 X(s) - s x(0) - \dot{x}(0)) + \gamma(s X(s) - x(0)) + k X(s) = \frac{1}{s - i\omega}$$

$$x(0) = \dot{x}(0) = 0$$

$$m(s^2 X(s)) + \gamma(s X(s)) + k X(s) = \frac{1}{s - i\omega}$$

$$X(s) [ms^2 + \gamma s + k] = \frac{1}{s - i\omega}$$

$$X(s) = \frac{1}{(s - i\omega)(ms^2 + \gamma s + k)}$$

$$\text{Substituting } \frac{\partial \mu}{\partial s} = \frac{\gamma}{m} \quad \text{and} \quad \cos^2 = \frac{k}{m}$$

$$X(s) = \frac{1}{(s - i\omega)m(s^2 + 2\mu s + k)}$$

$$= \frac{1}{(s - i\omega)m(s + \mu + \sqrt{\mu^2 - \omega_n^2})(s + \mu - \sqrt{\mu^2 - \omega_n^2})}$$

$$= \frac{1}{(s+a)(s+b)(s+c)}$$

$$a = -i\omega$$

$$b = \mu + \sqrt{\mu^2 - \omega_n^2}$$

$$c = \mu - \sqrt{\mu^2 - \omega_n^2}$$

(AMPAD)

Previous

Next

$$x(t) = \frac{e^{-i\omega t}}{\omega(-i\omega - \mu - \sqrt{\mu^2 - \omega_n^2})(-i\omega - \mu + \sqrt{\mu^2 + \omega_n^2})} - \frac{e^{(-\omega - \sqrt{\mu^2 - \omega_n^2})t}}{\omega(-i\omega - \mu - \sqrt{\mu^2 - \omega_n^2})(2\sqrt{\mu^2 - \omega_n^2})}$$

- $e^{(-\mu + \sqrt{\mu^2 - \omega_n^2})t}$ ~~gaseous~~

f)

$$m\ddot{x} + kx + \cancel{\epsilon}\dot{x} = 0$$

$$x = x_0 + \epsilon x_1 + \mathcal{O}(\epsilon^2)$$

$$\text{unperturbed: } m\ddot{x}_0 + kx_0 = 0$$

lowest order correction equation

$$m\ddot{x}_1 + kx_1 + \cancel{kx_0} = 0$$

↑
there should be something else
(with x_0) here

$$\text{remember, } x_0 = Ae^{i\omega t}$$

$$\text{↑ where } \omega = \sqrt{\frac{k}{m}}$$

solution to S.H.M.

$$\text{then } m\ddot{x}_1 + kx_1 + Ae^{i\omega t} = 0$$

$$\underline{m\ddot{x}_1 + kx_1 = -Ae^{i\omega t}}$$

problem! This is not a harmonic
as we'd expect it to be.

Equations of motion : $\underline{m}\ddot{\underline{x}}_1 + 2kx_1 - kx_2 = 0$

$$\underline{m}\ddot{\underline{x}}_2 + 2kx_2 - kx_1 = 0$$

$$\begin{bmatrix} \underline{m} & \underline{m} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{m}\ddot{\underline{x}} + \underline{k}\underline{x} = \underline{0}$$

$$|\underline{k} - \underline{\lambda} \underline{m}| = 0 \quad \text{for eigenvalues } \underline{\lambda}$$

$$\begin{vmatrix} 2k - \underline{\lambda}m & -\underline{\lambda}m \\ -\underline{\lambda}m & 2k - 2\underline{\lambda}m \end{vmatrix} = 4k^2 - 4k\underline{\lambda}m + \underline{\lambda}^2m^2 - k^2 = 0$$

$$3k^2 - 4k\underline{\lambda}m + \underline{\lambda}^2m^2 = 0$$

$$\underline{\lambda}_1 = \frac{3k}{m} \quad \underline{\lambda}_2 = \frac{k}{m}$$

$$(\underline{k} - \underline{\lambda}_1 \underline{m}) \vec{v}_1 = 0 \quad \text{for eigenvectors } \vec{v}_1$$

$$\begin{bmatrix} 2k - 3k & -k \\ -k & 2k - 3k \end{bmatrix} \vec{v}_1 = 0 \quad \vec{v}_1 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\begin{bmatrix} 2k - k & -k \\ -k & 2k - k \end{bmatrix} \vec{v}_2 = 0 \quad \vec{v}_2 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

From the MM book : $\ddot{\underline{x}} + \underline{A}\underline{x} = \underline{0}$

where $\underline{A} \cdot \underline{m} = \underline{m} \otimes$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{bmatrix}$$

$$\underline{m}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{bmatrix} \frac{3k}{m} & \frac{k}{m} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

where

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{Bmatrix}$$

$$\ddot{z}_1 + \frac{3k}{m} z_1 = 0$$

$$\ddot{z}_2 + \frac{k}{m} z_1 = 0$$

He caused us to expect solve it first

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

$$\text{let } x_1 = A e^{\lambda t} \quad x_2 = B e^{\lambda t}$$

$$A(mr^2 + 2kr) - Bk\cancel{r} = 0$$

$$B(mr^2 + 2kr) - Ak\cancel{r} = 0$$

$$\text{let } \frac{B}{A} = C$$

$$mr^2 + 2kr - Ck\cancel{r} = 0$$

$$C(mr^2 + 2kr) - k\cancel{r} = 0$$

$$r^2 = \frac{k}{m}(C-2)$$

$$\Rightarrow Ck(C-2) + 2kC - k = 0$$

$$C^2 = 1 \quad \omega_0 \quad C = \pm 1$$

$$r_1^2 = -\frac{k}{\omega} \quad r_2^2 = -\frac{3k}{\omega}$$

$$r_1 = -\frac{k}{\omega} i \quad r_2 = -\frac{3k}{\omega} i$$

$$\chi_1 = e^{-\frac{3k}{4\omega} it} \quad \chi_2 = e^{-\frac{k}{4\omega} it}$$

$$\chi_1 = e^{-\frac{k}{\omega} it} \quad \chi_2 = e^{-\frac{3k}{\omega} it}$$

(3)

$$y^{(k)} = \alpha y^{(k-1)} + (1-\alpha)x^{(k)}$$

$$Y(z) = \alpha z^{-1} Y(z) + y^{(k-1)} + (1-\alpha)x^{(k)}$$

$$Y(z) [1 - \alpha z^{-1}] = (1-\alpha)X(z)$$

$$Y(z) = \frac{1-\alpha}{1-\alpha z^{-1}} X(z)$$

$$Y(z) = (1-\alpha) \frac{z}{z-\alpha} X(z)$$

transform reversal:

$$y^{(k)} = (1-\alpha)x^{(k)}\alpha^k$$

$$\boxed{y^{(k)} = \alpha^k (1-\alpha)x^{(k)}}$$

$$h^{(k)} = \frac{y^{(k)}}{x^{(k)}} = \alpha^k (1-\alpha)$$

Response for arbitrary input (convolution w/ impulse response)
from ch3 notes

$$\begin{aligned} y^{(k)} &= \sum_{n=0}^k h^{(n)} x^{(k-n)} \\ &= \sum_{n=0}^k (1-\alpha)\alpha^n x^{(k-n)} \end{aligned}$$

frequency response, $y^{(k)} = e^{j\omega s+ k} H(e^{j\omega s})$

when $x^{(k)} = e^{j\omega s k}$

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$$H(e^{i\omega t}) = 1 - \alpha \frac{e^{i\omega t}}{e^{i\omega t} - \alpha}$$

$$|H(e^{i\omega t})| = \left| \begin{array}{cc} (1-\alpha) & e^{i\omega t} \\ i\omega t - \alpha & \end{array} \right|$$

$$= (1-\alpha) \left| \frac{1}{1 - \alpha e^{-i\omega t}} \right|$$

$$= \frac{1 - \alpha}{\sqrt{(1 - \alpha \cos \omega t)^2 + (\alpha \sin \omega t)^2}}$$

$$= \frac{1 - \alpha}{\sqrt{1 - 2\alpha \cos \omega t + \alpha^2 \cos^2 \omega t + \alpha^2 \sin^2 \omega t}}$$

$$= \frac{1 - \alpha}{\sqrt{1 - 2\alpha \cos \omega t + \alpha^2}}$$

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