2.1 a

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$$

What conditions will governing eqtn for small displacements of particle around arbitrary potential minima be simple undamped harmonic motion.

$$m\ddot{x} + \cancel{\gamma}\dot{x} + kx = e^{i\omega t}$$

$$m\ddot{x} + kx = e^{i\omega t}$$

$x \to 0$ Potential minimum, no stored energy

$$m\ddot{x} = e^{i\omega t}$$

$$\ddot{x} = \frac{e^{i\omega t}}{m}$$

let $\lambda = i\omega$

let $x = e^{\lambda t}$

$$\dot{x} = \lambda e^{\lambda t}$$

$$\ddot{x} = \lambda^2 e^{\lambda t}$$

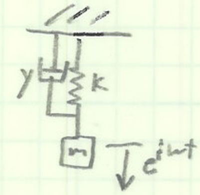
$$m\lambda^2 \cancel{e^{\lambda t}} = \cancel{e^{\lambda t}}$$

$$\lambda = \sqrt{\frac{1}{m}}$$

$$\boxed{x(t) = C_1 e^{\sqrt{\frac{1}{m}}t}}$$

2.1/ Consider motion of damped, driven harmonic oscillator.

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$$



b) Find solution to the homogeneous eqtn.
Comment on possible cases. How does Amplitude depend on frequency?

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

Assume constant coefficients

choose $x = e^{\lambda t}$
$\dot{x} = \lambda e^{\lambda t}$
$\ddot{x} = \lambda^2 e^{\lambda t}$

$$(m\lambda^2 + \gamma\lambda + k)\cancel{e^{\lambda t}} = 0$$

$$\lambda^2 + \frac{\gamma}{m}\lambda + \frac{k}{m} = 0$$

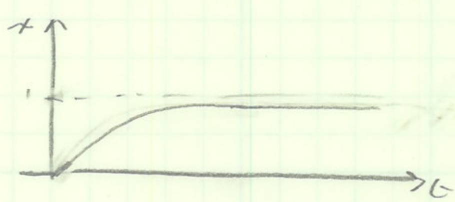
let $a = \frac{\gamma}{m}$
$b = \frac{k}{m}$

$$\lambda^2 + a\lambda + b = 0$$

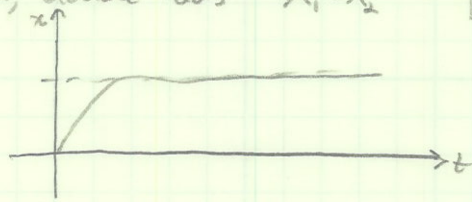
$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

<u>3 possible cases:</u>

I: Distinct, real roots λ_1, λ_2: $\boxed{a^2 > 4b}$ <u>overdamped</u>



II: Real, double roots $\lambda_1 = \lambda_2$ $\boxed{a^2 = 4b}$ <u>Critically damped</u>



III: Complex roots λ_1, λ_2 : $\boxed{a^2 < 4b}$ <u>underdamped</u>

$x_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Amplitude is dependent on frequency if there is a velocity term in initial conditions
ie
$x(0) = n$
$x'(0) = m$

$x(0) = n = c_1 + c_2 \Rightarrow \boxed{c_2 = n - c_1}$

$x'(0) = m = c_1\lambda_1 + c_2\lambda_2$

$m = c_1\lambda_1 + (n - c_1)\lambda_2$

$m = c_1(\lambda_1 - \lambda_2) + n\lambda_2$

$$\boxed{c_1 = \frac{m - n\lambda_2}{(\lambda_1 - \lambda_2)}}$$

2.1c Find particular solution to inhomogeneous problem by assuming response @ driving frequency.
Plot magnitude phase as function of driving frequency
$m = k = 1 \quad \gamma = 0.1$

$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t}$

Try $x_p(t) = Ce^{i\omega t}$

$\dot{x}_p(t) = i\omega C e^{i\omega t}$
$\ddot{x}_p(t) = i^2\omega^2 C e^{i\omega t} = -\omega^2 C e^{i\omega t}$

Need to check type of homogeneous problem first
$a = \frac{.1}{1} = 0.1$

$b = 1 \qquad a^2 < 4b$
$\qquad .01 < 4 \qquad$ underdamped

$$\lambda = \frac{-0.1 \pm \sqrt{.01 - 4}}{2} = -.005 \pm i\sqrt{3.99}$$

$x_h(t) = e^{-.005x}(A\cos\sqrt{3.99}x + B\sin\sqrt{3.99}x)$

Assume initial conditions $X(0) = 1 \quad X'(0) = 0$

$x_h(0) = \boxed{1 = A}$
$\dot{x}_h(t) = -.005e^{-.005x}(\cos\sqrt{3.99}x + B\sin\sqrt{3.99}x) + e^{-.005x}(\sqrt{3.99}\sin\sqrt{3.99x} + B\sqrt{3.99}\cos\sqrt{3.99x})$

$$\dot{x}_h(t) = e^{-.005x}\left[-.005\left(\cos\sqrt{3.99}x + \beta\sin\sqrt{3.99}x\right) - \sqrt{3.99}\left(\sin\sqrt{3.99}x + \beta\cos\sqrt{3.99}x\right)\right]$$

$$\dot{x}_h(0) = 0 = -.005(1) + \sqrt{3.99}(\beta)$$

$$\beta = \frac{.005}{\sqrt{3.99}} =$$

$$x_h(t) = e^{-.005x}\left(\cos\sqrt{3.99}x + \frac{.005}{\sqrt{3.99}}\sin\sqrt{3.99}x\right)$$

Now substitute particular solution

$$(-w^2C + aiwC + bC)e^{iwt} = e^{iwt}$$

$$C(-w^2 + aiw + b) = 1$$

$$C = \frac{1}{-w^2 + aiw + b}$$

$$x_p(t) = \frac{1}{-w^2 + aiw + b}e^{iwt}$$

$$\boxed{x(t) = e^{-.005t}\left(\cos\sqrt{3.99}t + \frac{.005}{\sqrt{3.99}}\sin\sqrt{3.99}t\right) + \frac{e^{iwt}}{-w^2 + 0.1iw + 1}}$$

Plot

2.1d/

Quality factor

homogeneous

$$Q = 2\pi \cdot \frac{\text{Stored Energy}}{\text{Dissipated Energy}} = 2\pi \cdot \frac{PE+KE}{\text{Damping}}$$

For mechanical system

$$Q = \frac{\sqrt{MK}}{D} = \frac{\sqrt{mk}}{\gamma}$$

Particular

$$Q = \frac{w_0}{\Delta w}$$ where w_0 = natural frequency

Δw = half power bandwidth (angular) where half power of system has been lost.

$$Q = \frac{1}{2\xi}$$

$$a = 2\xi w_0$$

$$b = w_0^2$$

$$w_0 = \sqrt{b} = \sqrt{\frac{k}{m}}$$

$$\xi = \frac{a}{2w_0} = \frac{\gamma/m}{2\sqrt{\frac{k}{m}}} \cdot \frac{\sqrt{\frac{k}{m}}}{\sqrt{\frac{k}{m}}}$$

$$= \frac{\gamma/\cancel{m} \cdot \frac{\sqrt{k}}{\sqrt{m}}}{2\frac{k}{\cancel{m}}} = \frac{\gamma \cdot \sqrt{k}}{2k\sqrt{m}}$$

$$Q = \frac{1}{\cancel{2}\frac{\gamma\sqrt{k}}{\cancel{2}k\sqrt{m}}} = \frac{k\sqrt{m}}{\gamma\sqrt{k}} = \frac{\sqrt{k}\cdot\cancel{\sqrt{k}}\sqrt{m}}{\gamma\cancel{\sqrt{k}}} = \frac{\sqrt{km}}{\gamma}$$

$$Q = \frac{\sqrt{mk}}{\gamma}$$

$$Q = 2\pi\frac{\left(\frac{1}{2}m\dot{x}^2 + m\ddot{x}x\right)}{\gamma} = \frac{2\pi}{\gamma}\left(\frac{1}{2}m\lambda^2 e^{2\lambda t} + \lambda^2 m e^{\lambda t}\cdot e^{\lambda t}\right)$$

$$= \frac{2\pi}{\gamma}\lambda^2 e^{2\lambda t}\left(\frac{3m}{2}\right) = \frac{3m\pi}{\gamma}\lambda^2 e^{2\lambda t}$$

$$Q_h = Q_p \Rightarrow \frac{\sqrt{mk}}{\gamma} = \frac{3m\pi\lambda^2 e^{2\lambda t}}{\gamma}$$?

$\boxed{2.1e}$ Solve using LaPlace transform

$$m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega t} \qquad x(0) = \dot{x}(0) = 0$$

$$\mathcal{L}\{\ddot{x}(t)\} = s^2 X(s)$$

$$ms^2 X(s) + \gamma s X(s) + kX(s) = \frac{1}{s - i\omega}$$

$$X(s)\left(s^2 + \frac{\gamma}{m}s + \frac{k}{m}\right) = \frac{1}{s - i\omega}$$

$$X(s) = \frac{1}{s^2 + \frac{\gamma}{m}s + \frac{k}{m}} \cdot \frac{1}{s - i\omega}$$

let $a = \frac{\gamma}{m}$, $b = \frac{k}{m}$

$a = .1$, $b = 1$

$$= \frac{1}{s^2 + as + b} \cdot \frac{1}{s - i\omega}$$

find roots $s^2 + as + b$

$$s = \frac{-a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2}$$

$$= -.05 \pm \frac{\sqrt{3.99}}{2}$$

$$X(s) = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2} \cdot \frac{1}{s - i\omega}$$

cos

const

Partial Fraction Expansion:

$$\frac{1}{\left((s+\frac{a}{2})^2 + b - \frac{a^2}{4}\right)(s - i\omega)} = \frac{A}{(s + \frac{1}{2}a) + b - \frac{1}{4}a^2} + \frac{B}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2} + \frac{C}{s - i\omega}$$

Solve C:

$$\frac{1}{(s+\frac{a}{2})^2 + b - \frac{a^2}{4}} = \frac{A\cancel{(s - i\omega)}}{(s+\frac{a}{2})^2 + b - \frac{a^2}{4}} + \frac{B\cancel{(s - i\omega)}}{(s+\frac{a}{2})^2 + b - \frac{a^2}{4}} + C \qquad \text{set } s = i\omega$$

$$\boxed{C = \frac{1}{(i\omega + \frac{a}{2})^2 + \frac{b - a^2}{4}}}$$

Solve B:

$$\frac{1}{s - i\omega} = \frac{A\left[(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2\right]}{(s + \frac{1}{2}a) + b - \frac{a^2}{4}} + B + \frac{C\left((s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2\right)}{s - i\omega}$$

set $s = -\frac{1}{2}a$

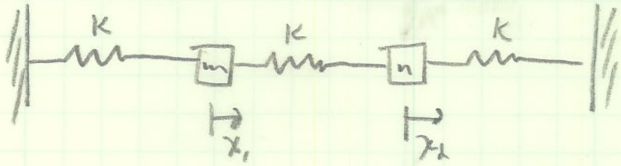
$$\frac{1}{-\frac{a}{2} - i\omega} = A + B + \frac{C\left(b - \frac{a^2}{4}\right)}{-\frac{a}{2} - i\omega}$$

⋮

not sure how to do this partial fraction it's blowing up...

2.1f see problem 2.1a

2.2



$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 = +kx_1 - 2kx_2$$

Set state variables

$$\left.\begin{aligned} y_1 &= x_1 \\ y_2 &= \dot{x}_1 \\ y_3 &= x_2 \\ y_4 &= \dot{x}_2 \end{aligned}\right\} y(t) \qquad y'(t) = \begin{aligned} \dot{y}_1 &= \dot{x}_1 \\ \dot{y}_2 &= \ddot{x}_1 \\ \dot{y}_3 &= \dot{x}_2 \\ \dot{y}_4 &= \ddot{x}_2 \end{aligned}$$

$$y'(t) = A\,y(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2k/m & 0 & k/m & 0 \\ 0 & 0 & 0 & 1 \\ k/m & 0 & -2k/m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

Calculate Eigenvalues $\det(A - \lambda I)$

$$\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ -2k/m & -\lambda & k/m & 0 \\ 0 & 0 & -\lambda & 1 \\ k/m & 0 & -2k/m & -\lambda \end{bmatrix} = \lambda^4 + \left(\frac{k}{m}\right)^2 = 0$$

$$\lambda^2 = \pm\frac{k}{m}$$

$$\boxed{\lambda = \pm\sqrt{\frac{k}{m}},\ \pm i\sqrt{\frac{k}{m}}}$$

2.3 Digital Smoothing Filter

$$y(k) = \alpha y(k-1) + (1-\alpha)x(k)$$

α = defines response of filter

Use z-transform to solve $y(k)$ assume $y(k<0)=0$
what is amplitude of freq response

Laplace 1st $y(k) - \alpha y(k-1) = (1-\alpha)x(k)$

$$Y(s) - \alpha e^{-as}Y(s) = (1-\alpha)X(s)$$

$$Y(s)(1 - \alpha e^{-as}) = (1-\alpha)X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{1-\alpha}{1-\alpha e^{-as}}$$

Convert to Z transfer

$$z = e^{as}$$

$$\frac{Y(z)}{X(z)} = \frac{1-\alpha}{1-\alpha z^{-k}}$$