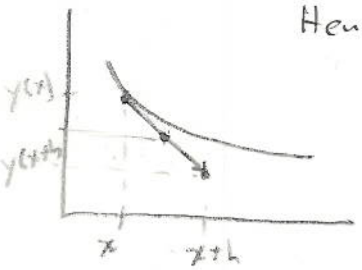
7.1 What is second order approximation error of Heun method, which averages slope at beginning & end of the interval



Heun:
$$y(x+h) = y(x) + \frac{h}{2}\left[f(x, y(x)) + f(x+h,\ y(x)+hf(x, y(x)))\right]$$

$$y(x+h) = y(x) + \frac{h}{2}$$

Evaluate error from the Taylor series. Expansion of the averaged step term.

$$\left[f(x, y(x)) + f(x+h,\ y(x)+hf(x,y(x)))\right] = f(x,y(x)) + f(x+h,\ y(x)+hf(x,y(x)))$$

$$\Rightarrow f\left[(x+h),\ y(x)+hf(x,y(x))\right] = f(x, y(x)) + h\frac{\partial}{\partial h}f(x+h,\ y(x)+hf(x,y(x)))$$

$$\text{where, } \frac{\partial f}{\partial h} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f(x,y(x))$$

notes: confusing because x, y get re-used

$$\Rightarrow f\left[(x+h),\ y(x)+hf(x,y(x))\right] = f(x,y(x)) + h\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f(x,y(x))\right]$$

$$\Rightarrow y(x+h) = y(x) + \frac{h}{2}\left\{f(x,y(x)) + f(x,y(x)) + h\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f(x,y(x))\right]\right\}$$

$$\boxed{y(x+h) = y(x) + hf(x,y(x)) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f(x,y(x))\right] + O(h^3)}$$

7.2) Harmonic oscillator $\ddot{y} + y = 0$ $y(0) = 1$, $\dot{y}(0) = 0$

Find: $y(t)$ $t = 0 \rightarrow 100\pi$

use: Euler method & 4th Order, fixed-step Runge-Kutta

Check how average local error, & error in final value & slope depend on step size.

First evaluate analytically so check error

$b^2 - 4ac = 1 - 4 = -3 < 0$

Solution is of the form $y(t) = A\cos t + B\sin t$ $y(0) = 1 = A\cos(0) + B\sin(0) \Rightarrow A = 1$

$y'(t) = -A\sin t + B\cos t$ $y'(0) = 0 = -\sin(0) + B\cos(0) \Rightarrow B = 0$

$\Rightarrow \boxed{y(t) = \cos t}$

Check: $y'(t) = -\sin t$

$y''(t) = -\cos t$

$\ddot{y} + y = 0 \Rightarrow -\cos t + \cos t = 0$ ✓

Euler method of Numerical Approximation:

$y(x+h) = y(x) + h f(x, y(x))$ where $\frac{dy}{dx} = f(x, y(x)) = -\sin t$

$x=0$ $y(0) = 1 + 1(-\sin(0)) = 1$

$x=1$ $y(1) = 1 + 1(-\sin(1))$

See website for more

7.3

write an adaptive stepper for 4th order Runge-Kutta

Check how average step size depends on local error

h_{nom} = nominal step size

$y(x+h) = \cdots$

full step = $y(x+h)$

half $step_1$ = $y(x+\frac{h}{2})$

half $step_2$ = $y(x+\frac{h}{2}+\frac{h}{2})$

error check = full step - half $step_2$

if error < min error

  h = h + Δ step size

if error > max error

  h = h - Δ step size

7.4 Numerically solve

acceleration of platform

$$l\ddot{\theta} + (g + \ddot{z})\sin\theta = 0$$

Take motion of platform as periodic. Interactively explore dynamics of the pendulum as function of amplitude + frequency of excitation

$x_1 = \theta$

$x_2 = \dot{x}_1 = \dot{\theta}$

$x_3 = \dot{x}_1 = x_2$

$x_4 = \dot{x}_2 = \ddot{x}_1 = \ddot{\theta}$

Input $z = A\sin\omega t$

$\dot{z} = A\omega\cos\omega t$

$\ddot{z} = -A\omega^2\sin\omega t$

$K = g + \ddot{z}$

$$l x_4 + (\underbrace{g + \ddot{z}}_{\text{constant?}})\sin x_1 = 0$$

$$l x_4 + K\sin x_1 = 0$$

$$f(x,y) = x_4 = \frac{-K}{l}\sin x_1$$

$$y(x+h) = y(x) + hf(x,y)$$

or

$x_1 = \theta$

$x_2 = \dot{x}_1 = \dot{\theta}$

$\dot{x}_2 = \ddot{x}_1 = \ddot{\theta}$

$$\Rightarrow l\dot{x}_2 + K\sin x_1 = 0$$

$$\Rightarrow \dot{x}_2 = \frac{-K}{l}\sin x_1 = f_2(x_1, x_2, t)$$

$$\dot{x}_1 = x_2 = f_1(x_1, x_2, t)$$

Equilibrium States:

$\dot{x}_1 = 0 = f_1(x_1, x_2, t)$

$\dot{x}_1 = 0 = x_2$

$\dot{x}_2 = 0 = \frac{-K}{l}\sin x_1 \Rightarrow x_1 = 0, \pi$

$\dot{x}_1 = x_2$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$