MAS864.14 $\checkmark$

8.1a) $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$ 1D wave Equation

Write down straight-forward finite-difference approximation

1^{st} order $\quad \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x} = \frac{\partial u}{\partial x}\Big|_{x,t} + O(\Delta x)$

2^{nd} order $\quad \frac{1}{\Delta x}\left[\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x} - \frac{u(x,t)-u(x-\Delta x,t)}{\Delta x}\right] = \frac{\partial^2 u}{\partial x^2}\Big|_{x,t} + O[(\Delta x)^2]$

$$\Rightarrow \left[\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}\right] = \frac{\partial^2 u}{\partial x^2}\Big|_{x,t} + O[(\Delta x)^2]$$

Using this forward difference method:

$$\boxed{\left[\frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{(\Delta t)^2}\right] = v^2\left[\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}\right] + O[(\Delta x)^2 + (\Delta t)^2]}$$

b) 1^{st} Order, error @ 2^{nd} order

c) Use von Neumann Stability Criteria to find mode amplitudes

Try the ansatz: $u_j^n = A(k)^n e^{ijk\Delta x}$ where n is time, j is space

$$\frac{\partial u}{\partial t} \Rightarrow \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{A^{n+1}e^{ijk\Delta x} - A^n e^{ijk\Delta x}}{\Delta t}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = \frac{A^{n+1}e^{ijk\Delta x} - 2A^n e^{ijk\Delta x} + A^{n-1}e^{ijk\Delta x}}{(\Delta t)^2}$$

$$\Rightarrow \frac{(A^{n+1} - 2A^n + A^{n-1})e^{ij\Delta x}}{(\Delta t)^2} = v^2 A^n \frac{\left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)}{(\Delta x)^2}$$

Divide by A^n

$$\Rightarrow (A^{-n})(A^{n+1} - 2A^n + A^{n-1})e^{ij\Delta x} = v^2\frac{(\Delta t)^2}{(\Delta x)^2}\left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)$$

$$(A - 2 + A^{-1}) = v^2\left(\frac{\Delta t}{\Delta x}\right)^2 e^{-ij\Delta x}\left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)$$

$$(A - 2 + A^{-1}) = v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

raise everything by an order

$$A^2 - 2A + 1 = v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(e^{i2k\Delta x} - 2e^{i2k\Delta x} + 1\right)$$

Try again,

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = v^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$$

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = v^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$u_j^{n+1} = \underbrace{2u_j^n - u_j^{n-1}}_{\text{time}} + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \underbrace{\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}_{\text{Space}}$$

Try ansatz from above: $u_j^n = A(k)^n e^{ijk\Delta x}$

$$A(k)^{n+1} e^{ijk\Delta x} = 2A(k)^n e^{ijk\Delta x} - A(k)^{n-1} e^{ijk\Delta x} + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \left(A(k)^n e^{i(j+1)k\Delta x} - 2A(k)^n e^{ijk\Delta x} + A(k)^n e^{i(j-1)k\Delta x}\right)$$

$$\Rightarrow A(k)^{n+1} e^{ijk\Delta x} = 2A(k)^n e^{ijk\Delta x} - A(k)^{n-1} e^{ijk\Delta x} + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 A(k)^n \left(e^{i(j+1)k\Delta x} - 2e^{ijk\Delta x} + e^{i(j-1)k\Delta x}\right)$$

Divide by ↑

$$A(k)^{n+1} = 2A(k)^n - A(k)^{n-1} + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 A(k)^n \left(\underbrace{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}\right)$$

$$e^{ik\Delta x} = \cos(k\Delta x) + i\sin(k\Delta x)$$

$$e^{-ik\Delta x} = \cos(k\Delta x) - i\sin(k\Delta x)$$

$$e^{i\theta} + e^{-i\theta} = \cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta} = 2\cos\theta$$

$$A(k)^{n+1} = -A(k)^{n-1} + A(k)^n\left(2 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2 \left(2\cos(ik\Delta x) - 2\right)\right)$$

let $n=1$

$$A(k)^2 = -1 + A(k)\left[2 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(2\cos(ik\Delta x) - 2\right)\right]$$

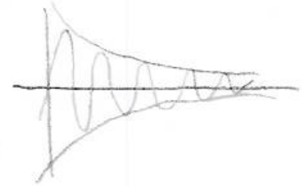
$$A(k)^2 - A(k)\underbrace{\left[2 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(2\cos(ik\Delta x) - 2\right)\right]}_{\beta} + 1 = 0$$

$$A(k)^2 - A(k)\beta + 1 = 0$$

$A(k) = \dfrac{\beta \pm \sqrt{\beta^2 - 4}}{2}$	where, $\beta = 2 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(2\cos(ik\Delta x) - 2\right)$

8.1 d) use $A(k)$ to find a condition of velocity, time step, space step for stability

For stability $A(k)$ should be damped
That is oscillation is okay but requires damping



$\beta^2 > 4$ over damped

$\beta^2 = 4$ critically damped

$\beta^2 < 4$ under damped ← potential instability

$\beta \geq 4$

$$4 \leq 2 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(2\cos(ik\Delta x) - 2\right) \Rightarrow 2 \leq 1 + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(\cos(ik\Delta x) - 1\right)$$

So each term needs to be addressed

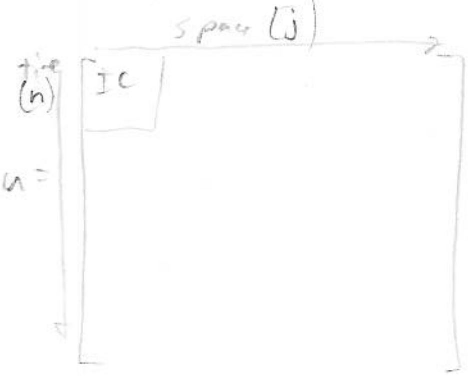
<u>Velocity:</u>
$$2 - 1 \leq v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(\cos(ik\Delta x) - 1\right)$$
$$\left(\frac{\Delta x}{\Delta t}\right)^2 \leq v^2\left(\cos(ik\Delta x) - 1\right)$$
$$\boxed{\frac{\Delta x}{\Delta t} \leq \pm v\sqrt{\cos(ik\Delta x) - 1}}$$

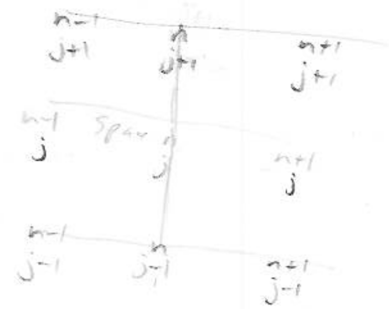
<u>Time:</u>
$$\boxed{\Delta t \geq \frac{\Delta x}{v\sqrt{\cos(ik\Delta x) - 1}}}$$

<u>Space:</u> not sure how to separate

e) Do different modes decay @ different rates?
yes the β term in $A(k)$ is dependent on k.

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + v^2\left(\frac{\Delta t}{\Delta x}\right)^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$





IC $\quad n=0 \quad j=0$

$\quad\quad n=1 \quad j=1$