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(3.1) In the absence of damping (i.e. $\gamma = 0$), the
(a) governing equation for small displacements of a particle around an arbitrary 1D potential minimum becomes

$m\ddot{x} + kx = e^{iwt}$. When the forcing function is zero

(i.e. $e^{iwt} = 0$), $m\ddot{x} = -kx$ which results in simple undamped harmonic motion

(b) Homogeneous equation : $m\ddot{x} + \gamma\dot{x} + kx = 0$

Let $x_g = Ce^{rt}$, $Ce^{rt}(mr^2 + \gamma r + k) = 0$

$$r_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}$$

If $\gamma^2 - 4mk > 0$, r_1 and r_2 both correspond to real roots and the system is overdamped

If $\gamma^2 - 4mk < 0$, r_1 and r_2 both correspond to complex roots, the system oscillates and is underdamped (e^{iwt})

If $\gamma^2 - 4mk = 0$, $r_1 = r_2$ and there is only one real root. The system is said to be critically damped

The general solution is $x_g = \sum_{n=1}^{N} C_n e^{r_n t}$

(3.1) For a particular solution, let $x_p = Ce^{i(wt-\phi)}$

(c) $Ce^{i(wt-\phi)}(-mw^2 + iw\gamma + k) = e^{iwt}$

$$C = \frac{e^{i\phi}}{k-mw^2+iw\gamma}$$

$$x_p = \frac{e^{iwt}}{k-mw^2+iw\gamma}$$

Magnitude, $|C| = \left[(k-mw^2)^2+(w\gamma)^2\right]^{-1/2}$, $e^{i\phi}$ is a constant

$$C(k-mw^2) + iCw\gamma = \cos\phi + i\sin\phi$$

$$\Rightarrow \tan\phi = \frac{Cw\gamma}{C(k-mw^2)}$$

Phase, $\phi = \tan^{-1}\left(\frac{w\gamma}{k-mw^2}\right)$

(d) Energy, $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{2}\gamma\dot{x}^2$

$$E \propto |C|^2 = \left[(k-mw^2)^2+(w\gamma)^2\right]^{-1}$$

For low damping, maximum energy occurs at $w = w_n = \sqrt{\frac{k}{m}}$

$$E_{max} \propto \left[\left(k-m\left(\tfrac{k}{m}\right)\right)^2 + \left(\tfrac{k}{m}\gamma^2\right)\right]^{-1} = \frac{m}{k\gamma^2}$$

Find width, $\frac{E_{max}}{2} = E$

$$\Rightarrow \frac{m}{2k\gamma^2} = \frac{1}{(k-mw^2)^2 + w^2\gamma^2}$$

$$\Rightarrow mk^2 - 2m^2w^2k + m^3w^4 + w^2\gamma^2 m = 2k\gamma^2$$

$$\Rightarrow m^3 w^4 + (\gamma^2 m - 2m^2 k) w^2 + mk^2 - 2k\gamma^2 = 0$$

$$w^2 = \frac{-(\gamma^2 m - 2m^2 k) \pm \sqrt{(\gamma^2 m - 2m^3 k)^2 - 4m^3(mk^2 - 2k\gamma^2)}}{2m^3}$$

$= \ldots$ Too much work, MATLAB to the rescue!

$$w_1 = \frac{\left[-\gamma\sqrt{km} + km\right]^{1/2}}{m}$$

where $\gamma^2, \gamma^3, \ldots \rightarrow 0$

$$w_2 = \frac{\left[\gamma\sqrt{km} + km\right]^{1/2}}{m}$$

Using Taylor Series Expansion about $\gamma = 0$

$$w_1 = \sqrt{\frac{k}{m}} - \frac{\gamma}{2m} + \ldots$$

$$w_2 = \sqrt{\frac{k}{m}} + \frac{\gamma}{2m} + \ldots$$

$$\text{Width} = \frac{w_2 - w_1}{2} = \frac{\gamma}{m}$$

Definition 1: Quality, $Q = w_n \cdot \frac{m}{\gamma}$

$$= \sqrt{\frac{k}{m}} \frac{m}{\gamma} = \frac{\sqrt{mk}}{\gamma}$$

$x_g = Ce^{rt}$, $\frac{t}{T} = \frac{1\,\text{rad}}{2\pi} \Rightarrow t = \frac{T}{2\pi} = \frac{1}{w_n} = \sqrt{\frac{m}{k}}$

When $t = 0$, $E_0 \propto C^2$

When $t = \sqrt{\frac{m}{k}}$, $E_1 \propto C^2 e^{2r\sqrt{\frac{m}{k}}} = C^2 e^{2\left(\frac{-\gamma}{2m}\right)\sqrt{\frac{m}{k}}} = C^2 e^{-\frac{\gamma}{\sqrt{mk}}}$

$$Q = \frac{E_0}{E_0 - E_1} = \frac{1}{1 - \frac{E_1}{E_0}} = \frac{1}{1 - e^{-\frac{\gamma}{\sqrt{mk}}}}$$

Using Taylor Series Expansion around $\gamma = 0$,

$$Q = -\frac{\gamma}{\sqrt{mk}} \quad ?$$

(3.1) (e)

$$\mathcal{L}\{m\ddot{x} + \gamma\dot{x} + kx = e^{iwt}\}$$

$$m[s^2X(s) - sx(0) - \dot{x}(0)] + \gamma[sX(s) - x(0)] + kX(s)$$

$$= \frac{1}{s - iw}$$

For $x(0) = \dot{x}(0) = 0$,

$$ms^2X(s) + \gamma sX(s) + kX(s) = \frac{1}{s-iw}$$

$$X(s) = \frac{1}{s-iw} \frac{1}{ms^2+\gamma s+k}$$

$$= \frac{1}{s-iw} \frac{1}{s-r_1} \frac{1}{s-r_2}$$

Using partial fractions,

$$X(s) = \frac{A}{s-iw} + \frac{B}{s-r_1} + \frac{C}{s-r_2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$= Ae^{iwt} + Be^{r_1 t} + Ce^{r_2 t}$$

(3.1) Let $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$

(8) and $m\ddot{x} + kx + \varepsilon x^2 = 0$

$$m\ddot{x}_0 + kx_0 + \varepsilon x_0^2 + m\varepsilon\ddot{x}_1 + k\varepsilon x_1 + \varepsilon^2 x_1^2 + \dots = 0$$

$\Rightarrow m\ddot{x}_0 + kx_0 = 0$

$\Rightarrow x_0^2 + m\ddot{x}_1 + kx_1 = 0$

where $x_0 = ?$

$\vdots$

$x_1 = \dots$

(3.2) $m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$

$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = +kx_1 - 2kx_2$

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} +2k & -k \\ -k & +2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad M\ddot{X} + KX = 0$$

$$\ddot{X} + M^{-1}KX = 0$$

$$M^{-1}K = \begin{bmatrix} \frac{2k}{m} & -k/m \\ -k/m & 2k/m \end{bmatrix}$$

Find eigenvalues, $\det\left[M^{-1}K - \lambda I\right] = 0$

$$\Rightarrow \det\begin{bmatrix} \frac{2k}{m} - \lambda & -k/m \\ -k/m & 2k/m - \lambda \end{bmatrix} = 0$$

$$\left(\frac{2k}{m} - \lambda\right)^2 - \left(\frac{-k}{m}\right)^2 = 0$$

$$\left(\frac{2k}{m} - \lambda_1 + \frac{k}{m}\right)\left(\frac{2k}{m} - \lambda_2 - \frac{k}{m}\right) = 0$$

$\Rightarrow \lambda_1 = \frac{3k}{m}, \quad \lambda_2 = \frac{k}{m}$

For $\lambda_1 = 3k/m$, the eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ which corresponds to both masses moving out of phase. For $\lambda_2 = \frac{k}{m}$, it is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where both masses are in phase

(3.3) $y(k) = \alpha y(k-1) + (1-\alpha)x(k)$

$$Y(z) = \alpha z^{-1}Y(z) + \cancel{y(-1)}^{\,0} + (1-\alpha)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\alpha}{1-\alpha z^{-1}} = (1-\alpha)\frac{z}{z-\alpha}$$

$$\Rightarrow h(k) = (1-\alpha)\alpha^k$$

$$y(k) = \sum_{n=0}^{k}(1-\alpha)\alpha^k x(k-n)$$

From (3.57), $y(k) = H(e^{j\omega\delta_t})e^{j\omega\delta_t k}$ when $x(k) = e^{j\omega\delta_t k}$

Amplitude, $|H(e^{j\omega\delta_t})| = \left|\frac{1-\alpha}{1-\alpha e^{-j\omega\delta_t}}\right|$

$$= \left|\frac{1-\alpha}{1-\alpha\cos(-\omega\delta_t) - \alpha j\sin(-\omega\delta_t)}\right|$$

$$= \frac{1-\alpha}{[(1-\alpha\cos(\omega\delta_t))^2 + \alpha\sin^2(\omega\delta_t)]^{1/2}}$$

$$= \frac{1-\alpha}{[1-2\alpha\cos(\omega\delta_t) + \alpha^2]^{1/2}}$$