

$$\begin{aligned}
 (6.1) \quad (a) \quad \log \langle e^{kx} \rangle &= \sum_{n=1}^{\infty} \frac{(k)^n}{n!} C_n \\
 &= \frac{k}{1!} C_1 + \frac{(k)^2}{2!} C_2 + \frac{(k)^3}{3!} C_3 + \dots \\
 &= kC_1 - \frac{1}{2}k^2 C_2 - \frac{1}{6}k^3 C_3 + \dots
 \end{aligned} \tag{6.24}$$

$$\exp \left(\sum_{n=1}^{\infty} \frac{(k)^n}{n!} C_n \right) = \sum_{n=0}^{\infty} \frac{(k)^n}{n!} \langle x^n \rangle \tag{6.25}$$

$$\begin{aligned}
 1 + (kC_1 - \frac{1}{2}k^2 C_2 - \frac{1}{6}k^3 C_3) + \frac{1}{2} (kC_1 - \frac{1}{2}k^2 C_2 - \frac{1}{6}k^3 C_3)^2 + \dots \\
 = 1 + k\langle x \rangle - \frac{1}{2}k^2 \langle x^2 \rangle - \frac{1}{6}k^3 \langle x^3 \rangle + \dots \\
 1 + kC_1 - \frac{1}{2}k^2 C_2 - \frac{1}{6}k^3 C_3 + \frac{1}{2} (-k^2 C_1^2 - k^3 C_1 C_2) + \frac{1}{6} (-k^3 C_1^3) + \dots \\
 = 1 + k\langle x \rangle - \frac{1}{2}k^2 \langle x^2 \rangle - \frac{1}{6}k^3 \langle x^3 \rangle + \dots
 \end{aligned}$$

Comparing coefficients of k ,

$$C_1 = \langle x \rangle$$

Comparing coefficients of k^2 ,

$$-\frac{1}{2}C_2 - \frac{1}{2}C_1^2 = -\frac{1}{2}\langle x^2 \rangle$$

$C_2 = \langle x^2 \rangle - \langle x \rangle^2$, this is the same as the variance

Comparing coefficients of k^3 ,

$$-\frac{1}{6}C_3 - \frac{1}{2}C_1 C_2 - \frac{1}{6}C_1^3 = -\frac{1}{6}\langle x^3 \rangle$$

$$\begin{aligned}
 C_3 &= \langle x^3 \rangle - 3C_1 C_2 - C_1^3 = \langle x^3 \rangle - 3\langle x \rangle (\langle x^2 \rangle - \langle x \rangle^2) - \langle x \rangle^3 \\
 &= \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 3\langle x \rangle^3 - \langle x \rangle^3 \\
 &= \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 2\langle x \rangle^3
 \end{aligned}$$

$$(6.1) \quad \text{For } p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\bar{x})^2/2\sigma^2}$$

$$(b) \quad \langle x \rangle = \int_{-\infty}^{\infty} x p(x) dx = \bar{x} \quad (\text{Mean value}) \quad (6.4)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x) dx = \sigma^2 + \langle x \rangle^2 = \sigma^2 + \bar{x}^2 \quad (6.5)$$

$$\langle x^3 \rangle = \int_{-\infty}^{\infty} x^3 p(x) dx = ?$$

$$\begin{aligned}\sigma^3 &= \langle (x-\bar{x})^3 \rangle = \langle (x-\bar{x})(x^2 - 2x\bar{x} + \bar{x}^2) \rangle \\&= \langle (x^3 - 2x^2\bar{x} + x\bar{x}^2 - \bar{x}x^2 + 2x\bar{x}^2 - \bar{x}^3) \rangle \\&= \langle x^3 \rangle - 3\langle x^2 \rangle \bar{x} + 3\langle x \rangle \bar{x}^2 - \bar{x}^3 \\&= \langle x^3 \rangle - 3(\sigma^2 + \bar{x}^2)\bar{x} + 3\bar{x}^3 - \bar{x}^3 \\&= \langle x^3 \rangle - 3\sigma^2\bar{x} - 3\bar{x}^3 + 3\bar{x}^3 - \bar{x}^3\end{aligned}$$

$$\Rightarrow \langle x^3 \rangle = 3\sigma^2\bar{x} + \bar{x}^3 + \sigma^3$$

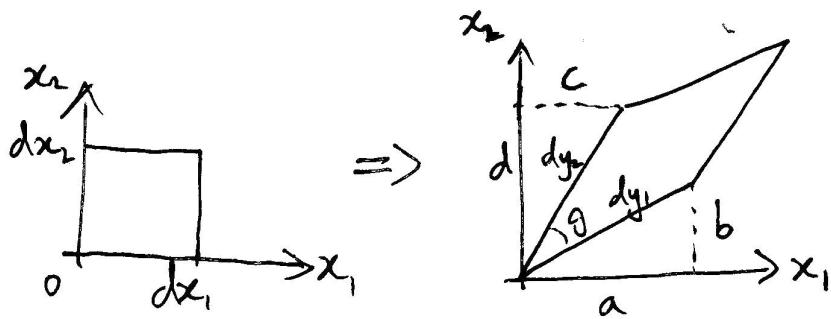
$$C_1 = \langle x \rangle = \bar{x}$$

$$\begin{aligned}C_2 &= \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2 + \bar{x}^2 - \bar{x}^2 \\&= \sigma^2\end{aligned}$$

$$\begin{aligned}C_3 &= \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 2\langle x \rangle^3 \\&= 3\sigma^2\bar{x} + \bar{x}^3 + \sigma^3 - 3\bar{x}(\sigma^2 + \bar{x}^2) + 2\bar{x}^3 \\&= 3\sigma^2\bar{x} + \bar{x}^3 + \sigma^3 - 3\bar{x}\sigma^2 - 3\bar{x}^3 + 2\bar{x}^3 \\&= \sigma^3\end{aligned}$$

$$(6.2) \quad \vec{y}(\vec{x}) = \begin{bmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{bmatrix}$$

(a)



$$\text{where } a = \frac{\delta y_1}{\delta x_1} dx_1, \quad b = \frac{\delta y_1}{\delta x_2} dx_2$$

$$c = \frac{\delta y_2}{\delta x_1} dx_1, \quad d = \frac{\delta y_2}{\delta x_2} dx_2$$

$$\text{Area of a parallelogram} = |\vec{dy}_1 \times \vec{dy}_2|$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} = |(0)\hat{i} + (0)\hat{j} + (ad-bc)\hat{k}|$$

$$= ad - bc$$

$$= \underbrace{\left(\frac{\delta y_1}{\delta x_1} \frac{\delta y_2}{\delta x_2} - \frac{\delta y_1}{\delta x_2} \frac{\delta y_2}{\delta x_1} \right)}_{\text{determinant of Jacobian, } J} \underbrace{dx_1 dx_2}_{\text{original area}}$$

$$\text{where } J = \begin{bmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_1}{\delta x_2} \\ \frac{\delta y_2}{\delta x_1} & \frac{\delta y_2}{\delta x_2} \end{bmatrix}$$

(6.2)
(b)

$$y_1 = \sqrt{-2 \ln x_1} \sin(x_2), \quad y_2 = \sqrt{-2 \ln x_1} \cos(x_2)$$

$$J = \begin{bmatrix} \frac{-\sin(x_2)}{x_1 \sqrt{-2 \ln x_1}} & \cos(x_2) \sqrt{-2 \ln x_1} \\ \frac{-\cos(x_2)}{x_1 \sqrt{-2 \ln x_1}} & -\sin(x_2) \sqrt{-2 \ln x_1} \end{bmatrix}$$

$$\det(J) = \left(\frac{\sin^2(x_2)}{x_1} - \left(\frac{-\cos^2(x_2)}{x_1} \right) \right) = \frac{1}{x_1}$$

$$p(y_1, y_2) = \left| \frac{1}{x_1} \right| p(x_1, x_2) \quad (\text{pg. 60})$$

(6.3) For an order 4 maximal LFSR,

(a) $x_n = x_{n-1} + x_{n-4} \pmod{2}$ (6.76)

| $n=0$ | | | | | | | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|---------|-------|-------|-------|-------|-------|---------|
| x_{-4} | x_{-3} | x_{-2} | x_{-1} | x_0 | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 | x_7 | x_8 | \dots |
| 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | |
| x_9 | x_{10} | x_{11} | x_{12} | x_{13} | x_{14} | x_{15} | \dots | | | | | | |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | | | | | | | |

(b) $1 \text{ GHz} = 10^9 \text{ s}^{-1}$

$$(2^M - 1) \times 10^{-9} \text{ s} = 10^{10} \text{ years} \times \left(\frac{365 \text{ days}}{1 \text{ year}} \right) \times \left(\frac{24 \text{ hours}}{1 \text{ day}} \right) \times \left(\frac{3600 \text{ s}}{1 \text{ hour}} \right)$$

$$2^M - 1 = 10^{19} \text{ years} \times 31536000 \text{ s/year}$$

$$M \cong 88$$

(6.4) $\frac{\delta P}{\delta t} = D \frac{\delta^2 P}{\delta x^2}$ (6.57)

(a) $\int \left\{ \frac{\delta P(x, t)}{\delta x} \right\} dx = P(k, t) = \int_{-\infty}^{\infty} e^{-ikx} P(x, t) dx$

$$P(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} P(k, t) dk, \text{ where } k \text{ is the wave number}$$

$$\frac{\delta P(k, t)}{\delta t} = (ik)^2 D P(k, t) = -k^2 D P(k, t)$$

$$\Rightarrow P(k, t) = C e^{-k^2 D t}$$

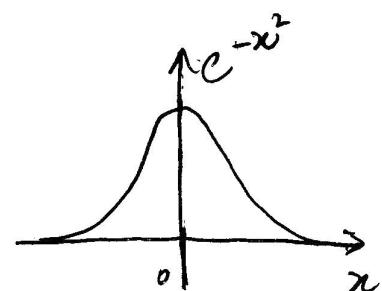
$$\Rightarrow P(x, t) = \frac{C_1}{2\pi} \int_{-\infty}^{\infty} e^{-k^2 D t + ikx} dk = \frac{C_1}{2\pi} e^{\frac{i^2 x^2}{4D t}} \int_{-\infty}^{\infty} e^{-(k\sqrt{D t} - \frac{ix}{2\sqrt{D t}})^2} dk$$

$$(6.4) \Rightarrow p(x,t) = \frac{C_1}{2\pi} e^{-\frac{x^2}{4Dt}} \int_{-\infty}^{\infty} e^{-Dt(k-\frac{1}{Dt})^2} dk$$

constant

$$= Ce^{-\frac{x^2}{4Dt}}$$

$$= Ce^{-\frac{x^2}{2\sigma^2}}$$



$$(b) \Rightarrow \sigma^2 = 2Dt \text{ is the variance}$$

$$\Rightarrow p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

$$(c) \text{ Since } \langle x \rangle = \bar{x} = 0,$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle = 2Dt$$

$$\langle x^2 \rangle = \frac{kTt}{3\pi Ma} \quad (6.71)$$

$$\Rightarrow D = \frac{kT}{6\pi Ma} \quad \text{where } a \text{ is the diameter of the particle}$$

μ is the viscosity of the fluid

$$(e) \int_{-\frac{3\sigma}{2}}^{\frac{3\sigma}{2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx = 0.8664 \quad (\text{from Z Table, } Z = 1.5)$$

$\bar{x} = 0$

is the fraction of trajectories that should be contained in the error bars