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(9.1)

(a) $$\frac{\delta^2 u}{\delta t^2} = v^2 \frac{\delta^2 u}{\delta x^2} + \gamma \frac{\delta}{\delta t}\frac{\delta^2 u}{\delta x^2}$$

Over the interval $[0,1]$, the Galerkin expansion is,

$$\int_0^1 \left[\frac{\delta^2 u}{\delta t^2} - v^2\frac{\delta^2 u}{\delta x^2} - \gamma \frac{\delta}{\delta t}\frac{\delta^2 u}{\delta x^2}\right]\phi_j(x)\,dx = 0 \qquad [\text{Source term} = 0]$$

Let $u(x,t) = \sum_i a_i(t)\phi_i(x)$,

$$\sum_i \int_0^1 \left(\frac{d^2 a_i}{dt^2}\phi_i\phi_j - v^2 a_i \phi_j \frac{d^2\phi_i}{dx^2} - \gamma\frac{da_i}{dt}\phi_j\frac{d^2\phi_i}{dx^2}\right)dx = 0$$

$$\Rightarrow \sum_i \frac{d^2 a_i}{dt^2}\underbrace{\int_0^1 \phi_i\phi_j\,dx}_{A_{ij}} - \left(v^2 a_i + \gamma\frac{da_i}{dt}\right)\int_0^1 \phi_j\frac{d^2\phi_i}{dx^2}\,dx = 0$$

$$\int_0^1 \phi_j\frac{d^2\phi_i}{dx^2}\,dx = \underbrace{\phi_j\frac{d\phi_i}{dx}\Big|_0^1 - \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}\,dx}_{-B_{ij}}$$

from (9.25)

$$\Rightarrow A\cdot\frac{d^2\vec{a}}{dt^2} + v^2 B\vec{a} + \gamma B\cdot\frac{d\vec{a}}{dt} = 0$$

is a system of differential equations to approximate the wave equation.

(b) $$A_{ij} = \int \phi_i\phi_j\,dx$$

For $i = j$,

$$A_{ii} = \int_{x_{i-1}}^{x_i}\left[\frac{x - x_{i-1}}{x_i - x_{i-1}}\right]^2 dx + \int_{x_i}^{x_{i+1}}\left[\frac{x_{i+1} - x}{x_{i+1} - x_i}\right]^2 dx$$

$$= \int_{x_i - h}^{x_i}\left[\frac{x - (x_i - h)}{h}\right]^2 dx + \int_{x_i}^{x_i + h}\left[\frac{(x_i + h) - x}{h}\right]^2 dx$$

$$= -\frac{[(x_i - h) - x]^3}{3h^2}\Big|_{x_i - h}^{x_i} - \frac{[x - (x_i + h)]^3}{3h^2}\Big|_{x_i}^{x_i + h} = \frac{2h}{3}$$

(9.1)
(b)
Continued

For $i \neq j$,

$$A_{ij} = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x_i - x}{x_i - x_{i-1}}\right) dx$$

$$= \int_{x_i - h}^{x_i} \left(\frac{x - (x_i - h)}{h}\right)\left(\frac{x_i - x}{h}\right) dx$$

$$= \left. \frac{\frac{1}{2}x^2(x_i - h + x_i) - (x_i - h)x_i x - x^3/3}{h^2} \right|_{x_i - h}^{x_i}$$

$$= \frac{-(x_i - h - x_i)^3}{6h^2} = \frac{h}{6}$$

$$B_{ij} = \int \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\, dx - \phi_j \frac{d\phi_i}{dx}\Big|$$

For $i = j$,

$$B_{ii} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{x_i - x_{i-1}}\right)^2 dx - \left.\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{1}{x_i - x_{i-1}}\right)\right|_{x_{i-1}}^{x_i}$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{-1}{x_{i+1} - x_i}\right)^2 dx - \left.\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{-1}{x_{i+1} - x_i}\right)\right|_{x_i}^{x_{i+1}}$$

$$= \int_{x_i - h}^{x_i} \frac{1}{h^2}\, dx - \left.\left(\frac{x - (x_i - h)}{h^2}\right)\right|_{x_i - h}^{x_i} + \int_{x_i}^{x_i + h} \frac{1}{h^2}\, dx - \left.\left(\frac{(x_i + h) - x}{h^2}\right)\right|_{x_i}^{x_i + h}$$

$$= \frac{1}{h} - \frac{1}{h} + \frac{1}{h} - \left(-\frac{1}{h}\right) = \frac{2}{h}$$

(9.1) For $i \neq j$,
(b)
Continued 2

$$B_{ij} = \int_{x_{j-1}}^{x_j} \left(\frac{1}{x_j - x_{j-1}}\right)\left(\frac{-1}{x_j - x_{j-1}}\right) dx - \left(\frac{x - x_{j-1}}{x_j - x_{j-1}}\right)\left(\frac{-1}{x_j - x_{j-1}}\right)\Bigg|_{x_{j-1}}^{x_j}$$

$$= \int_{x_j - h}^{x_j} \frac{-1}{h^2}\, dx + \left(\frac{x - (x_j - h)}{h^2}\right)\Bigg|_{x_j - h}^{x_j}$$

$$= \frac{-1}{h} + \frac{1}{h} = 0$$

(c) From (9.21),

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h & h^2 & h^3 \\ 0 & 1 & 2h & 3h^2 \end{bmatrix}^{-1} \begin{bmatrix} u_0 \\ \dot{u}_0 \\ u_h \\ \dot{u}_h \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} u_0 \\ \dot{u}_0 \\ u_h \\ \dot{u}_h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $u = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ Basis functions

(9.2) Potential Energy, $V = \int_0^L \left(\frac{1}{2} EI \left(\frac{d^2 u}{dx^2}\right)^2 - u(x) f(x)\right) dx$ (9.29)

$$u(\vec{x}, t) = \sum_{i}^{4} a_i(t)\, \phi_i(\vec{x}) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$\vec{a} = A^{-1} \cdot \vec{b}$ where (9.33)

$A_{ij} = \int_0^L EI \frac{d^2 \phi_i}{dx^2} \frac{d^2 \phi_j}{dx^2}\, dx$ [Stiffness matrix]

$\vec{b}_j = \int_0^L \phi_j(x) f(x)\, dx$ [Force vector]