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3.8 Problems

3.1 (a) no dissipation: $\gamma = 0$

no driving: homogeneous (not necessarily)

choose $x = e^{\Omega t}$

always exponential decay of transient.

$$m\Omega^2 + \gamma\Omega + k = 0 \rightarrow \Omega_{1,2} = \frac{-\gamma/m \pm \sqrt{\gamma^2/m^2 - 4\cancel{m}k/m}}{2} = -\frac{\gamma}{2m} \pm \underbrace{\sqrt{\frac{\gamma^2}{4m^2} - \cancel{4m}k/m}}_{\text{related to } \omega_0}$$

assume response at driving freq. → ansatz $x = e^{i\omega t}$. $\frac{\gamma^2}{4m^2} - \cancel{m}k < 0 \rightarrow \gamma^2 < 4m^{\cancel{2}}k$

$-m\omega^2 + i\gamma\omega + k = 1$ (constraints on $m, \gamma, k$).

amplitude decoupled from frequency. - just depends on the initial conditions. and on the damping.

$$U(x) \approx U(0) + \underbrace{U'(0)}_{\text{zero.}}x + \frac{1}{2}U''(0)x^2 + \cdots + \frac{1}{6}U'''(0)x^3 + \cdots$$

higher order terms far less relevant.

$$y(t) = c_1 e^{-t/\tau + i\omega_0 t} + c_2 e^{i\omega t}.$$

$$\dot{y}(t) = c_1\left(-\frac{1}{\tau} + i\omega_0\right)e^{-t/\tau + i\omega_0 t} + c_2 i\omega e^{i\omega t}.$$

$$\ddot{y}(t) = c_1\left(-\frac{1}{\tau} + i\omega_0\right)^2 e^{-t/\tau + i\omega_0 t} - c_2\omega^2 e^{i\omega t}$$

$$m\left[\ \text{-''-}\ \right] + \gamma\left[\ \text{-''-}\ \right] + k\,y(t) = e^{i\omega t}.$$

$$m\ddot{x} = -kx \qquad e^{\Omega t}: \qquad m\Omega^2 = -k \quad \Omega = \pm\sqrt{\frac{-k}{m}} = \pm i\omega.$$

i.e. rotation in either direction.

(c). $m\ddot{x} + \gamma\dot{x} + kx = e^{i\omega_d t}$

ansatz: $x(t) = Ce^{i\omega_d t} \longrightarrow \left[-m\omega_d^2 + i\gamma\omega_d + k\right]C = 1$

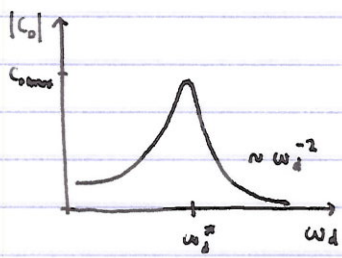
$$C = \frac{1}{k - m\omega_d^2 + i\gamma\omega_d}$$

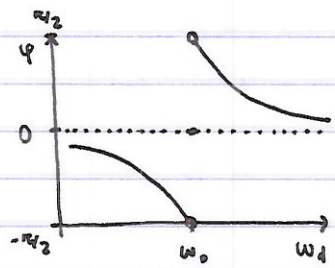
$$mC = \frac{1}{\omega_0^2 - \omega_d^2 + i\frac{\gamma}{m}\omega_d} \quad \text{with } \omega_0 = \sqrt{k/m}\ .$$

define $C_0 = mC$. then

$$|C_0| = \frac{1}{\sqrt{(\omega_d^2 - \omega_0^2)^2 + \left(\frac{\gamma\omega_d}{m}\right)^2}} \qquad \varphi = \tan^{-1}\left(\frac{-\gamma\omega_d/m}{\omega_0^2 - \omega_d^2}\right) \qquad \tan^{-1}(0) = 0$$

$$\tan^{-1}(\infty) = \pi/2$$




$$\frac{\partial |C_0|}{\partial \omega_d} = -\frac{1}{2}(\text{denom.})^{-3/2} \cdot \left[2(\omega_d^2 - \omega_0^2)\cdot 2\omega_d + \frac{2\gamma^2\omega_d}{m^2}\right] = 0$$

$$4\omega_d(\omega_d^2 - \omega_0^2) + \frac{2\gamma^2\omega_d}{m^2} = 0$$

nontrivial solution

$$\omega_d^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}$$

$$\omega_d^* = \pm\sqrt{\omega_0^2 - \frac{\gamma^2}{2m^2}}$$

$$|C_0|_{max} = |C_0|(\omega_d^*)$$

$$\left(\omega_0^2 - \frac{\gamma^2}{2m^2} - \omega_0^2\right)^2 + \left(\frac{\gamma}{m}\right)^2\left(\omega_0^2 - \frac{\gamma^2}{2m^2}\right) = \left(\frac{\gamma}{m}\right)^2\left[\omega_0^2 - \frac{\gamma^2}{2m^2} + \frac{\gamma^2}{4m^2}\right]$$

$$|C_0|_{max} = \frac{1}{\sqrt{\left(\frac{\gamma}{m}\right)^2\left[\omega_0^2 - \frac{\gamma^2}{4m^2}\right]}}$$

$|C_0|_{max}$ diverges when $\gamma \to 0$.

<u>resonance</u>

for $m = k = 1$ and $\gamma = 0.1$

$\omega_d^* \approx 1$ and $|C_0|_{max} \approx 10$.

(d). $Q = \dfrac{\omega_c}{FWHM}$. $\quad \langle E \rangle = \langle U + T \rangle = \left\langle \frac{1}{2} k x^2 + \frac{1}{2} m \dot{x}^2 \right\rangle$

for a driven system, $x(t) = C e^{i\omega_d t}$

$$\langle x(t)^2 \rangle = \frac{1}{2} C^2 \qquad \langle \dot{x}(t)^2 \rangle = \frac{1}{2} \omega_d^2 C^2 .$$

$$\langle E \rangle = \frac{1}{4} k C^2 + \frac{1}{4} m \omega_d^2 C^2$$

$$\langle E \rangle = \frac{1}{4} (k + m\omega_d^2) \frac{1}{(\omega_d^2 - \omega_0^2)^2 + \left(\frac{\gamma \omega_d}{m}\right)^2} = \frac{m}{4} \frac{\omega_d^2 + \omega_0^2}{(\omega_d^2 - \omega_0^2)^2 + \left(\frac{\gamma\omega_d}{m}\right)^2}$$

$\langle E \rangle_{max}$ via $\dfrac{\partial \langle E \rangle}{\partial \omega_d} = \dfrac{m}{4} \dfrac{(\text{denom}) \cdot 2\omega_d - (\omega_d^2 + \omega_0^2)\left(2(\omega_d^2 - \omega_0^2) 2\omega_d + 2\gamma^2 \omega_d / m^2\right)}{(\text{denom})^2} = 0$

$$2\omega_d \left[ (\omega_d^2 - \omega_0^2)^2 + \left(\frac{\gamma \omega_d}{m}\right)^2 \right] - (\omega_d^2 + \omega_0^2)\left[ 4\omega_d (\omega_d^2 - \omega_0^2) + \frac{2\gamma^2 \omega_d}{m^2} \right] = 0 \qquad - \frac{2\gamma^2}{m^2} \omega_d \omega_0^2$$

$$0 = 2\cancel{\omega_d}^5 - 4\omega_d^3\cancel{\omega_0}^2 + 2\cancel{\omega_d}\omega_0^4 + \frac{2\gamma^2}{\cancel{m^2}}\omega_d^3 - 4\cancel{\omega_d}^5 + 4\omega_d^3\cancel{\omega_0}^2 - 4\omega_d^3\cancel{\omega_0}^2 + 4\omega_d\cancel{\omega_0}^4 - \cancel{\frac{2\gamma^2}{m^2}}\omega_d^3 \wedge$$

$$-2\omega_d^5 - 4\omega_d^3\omega_0^2 + 6\omega_d\omega_0^4 - \frac{2\gamma^2}{m^2}\omega_d\omega_0^2 = 0$$

nontrivial

$$\omega_d^4 + 2\omega_d^2\omega_0^2 - 3\omega_0^4 + \frac{2\gamma^2}{m^2}\omega_0^2 = 0$$

$$\omega_d^{*2} = -\omega_0^2 \pm \sqrt{\omega_0^4 - \frac{2\gamma^2}{m^2}\omega_0^2 + 3\omega_0^4}$$

minus sign is unphysical.

$$\sqrt{A + Bx} \approx \sqrt{A} + \frac{1}{2}(A + Bx)^{-1/2} \cdot B \Big|_{x=0} \cdot x + O(x^2)$$

$$= \sqrt{A} + \frac{B}{2\sqrt{A}} x \qquad \text{let } A = 4\omega_0^4 \text{ and } B = -\frac{2\omega_0^2}{m^2}$$

then $\omega_d^{*2} \approx -\omega_0^2 + 2\omega_0^2 - \dfrac{2\omega_0^2}{2m^2 \cdot 2\omega_0^2}\gamma^2 = \omega_0^2 - \dfrac{\gamma^2}{2m^2}$ ← this is the same result we obtained above.

$$\langle E \rangle_{max} = \frac{m}{4} \cdot \frac{2\omega_0^2 - \gamma^2/2m^2}{\frac{\gamma^4}{4m^4} + \left(\frac{\gamma^2}{m^2}\right)\left(\omega_0^2 - \frac{\gamma^2}{2m^2}\right)} = \frac{2\left(\omega_0^2 - \gamma^2/4m^2\right)}{\left(\frac{\gamma}{m}\right)^2 \left(\omega_0^2 - \gamma^2/4m^2\right)} = \frac{2m^2}{\gamma^2}$$

as $\gamma \to 0$ the energy diverges

must find $\omega_d$ such that $\langle E \rangle = \frac{\langle E \rangle_{max}}{2} = \frac{m^2}{\gamma^2}$

$$\frac{m^2}{\gamma^2} = \frac{m}{4} \frac{\omega_d^2 + \omega_0^2}{(\omega_d^2 - \omega_0^2)^2 + \left(\frac{\gamma \omega_d}{m}\right)^2}$$

for an undriven oscillator, $Q = \frac{\text{energy}}{\text{energy lost/radian}}$.

since $x(t) \propto e^{-\gamma/2m\, t}$ the energy $\sim e^{-\gamma/m\, t}$.

$$\frac{d(\text{amplitude})}{dt} = -\frac{\gamma}{2m} e^{-\gamma/2m\, t}$$

so the energy lost/radian is $+\frac{\gamma^2}{4m^2} e^{-\gamma/m\, t}$.

$$\rightarrow Q = \frac{4m^2}{\gamma^2}$$

(e). Laplace transforms

$$m\ddot{x} + \gamma\dot{x} + kx \longrightarrow m\left(s^2 F(s) - sf(0) - \frac{df(0)}{dt}\right) + \gamma\left(sF(s) - f(0)\right) + kF(s)$$

$$ms^2 F(s) + \gamma s F(s) + kF(s) = \mathcal{L}\{e^{i\omega t}\} = \frac{1}{s - i\omega}$$

$$F(s) = \frac{1}{(s-i\omega)(ms^2 + \gamma s + k)}$$

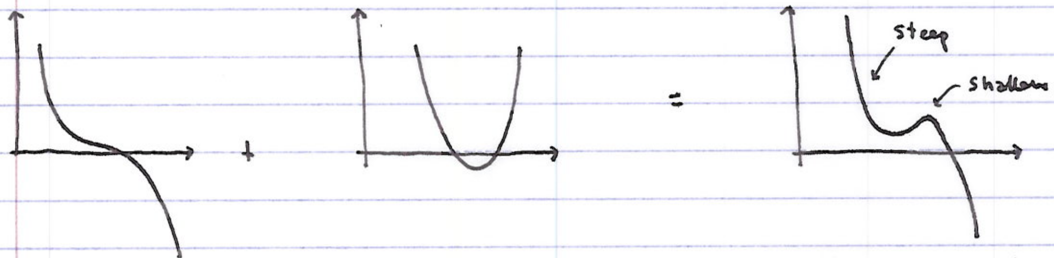
$$= \frac{1}{m(s-i\omega)(s-s_1)(s-s_2)}$$

$$\text{with } s_1, s_2 = \frac{-\gamma/m \pm \sqrt{(\gamma/m)^2 - 4k/m}}{2}.$$

from the inverse Laplace we obtain

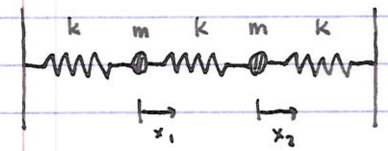
$$\frac{e^{-i\omega t}}{(-i\omega - s_1)(-i\omega - s_2)} - \frac{e^{s_1 t}}{(-i\omega - s_1)(s_1 - s_2)} - \frac{e^{s_2 t}}{(-i\omega - s_2)(s_2 - s_1)}.$$

(f). depending on the coefficients we could have a third order correction,
but this would not be globally stable:




- the 3rd order correction introduces some asymmetry.

(3.2)




$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$
$$\rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

find $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^{i\omega t}$ (normal mode).

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}(-\omega^2)\, e^{i\omega t}$$

for nontrivial solution we need determinant of $\begin{pmatrix} -2-\omega^2 & 1 \\ 1 & -2-\omega^2 \end{pmatrix} = 0$

$$(4 + 4\omega^2 + \omega^4) - 1 \rightarrow \omega^2 = 1, 3.$$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 1\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rightarrow \alpha_1, \alpha_2 = \frac{1}{\sqrt{2}} \quad \text{(symmetric)}$$

$$\omega_s = \sqrt{k/m}$$

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 3\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rightarrow \alpha_1 : +\frac{1}{\sqrt{2}},\ \alpha_2 : -\frac{1}{\sqrt{2}} \quad \text{(antisymmetric)}$$

$$\omega_s = \sqrt{\frac{3k}{m}}.$$

(3.3) $y(k) = \alpha y(k-1) + (1-\alpha)x(k)$

$$\mathcal{Z}\{y(k)\} = \mathcal{Z}\{\alpha y(k-1) + (1-\alpha)x(k)\}$$

$$Y(z) = \alpha z^{-1} Y(z) + (1-\alpha)\mathcal{Z}\{x(k)\}$$

$$(\alpha-1)X(z) = (\alpha z^{-1} - 1)Y(z)$$

$$Y(z) = X(z) \cdot \frac{\alpha - 1}{\alpha z^{-1} - 1} = (\alpha-1)X(z) \cdot \frac{z}{\alpha - z}$$

$\mathcal{Z}$ (convolution) = product

$$y(k) = \sum_{n=0}^{k} x(k-n) \cdot (1-\alpha)\alpha^{n}.$$

if $\alpha < 1$ then takes low frequency components.