ODE HW

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3.1

a)

In the event of simple undamped harmonic motion, physically no amount of kinetic or potential energy may escape the system in the form of friction, collisions, conversion to mass, etc. In a mathematical sense, for small displacements around a potential minimum at x_0, this occurs when there is no damping; when $\gamma = 0$.

b)

The homogenous equation

$$m\ddot{x} + \gamma\dot{x} + kx = 0 \tag{1}$$

can be solved by first predicting a solution of the form

$$x = e^{rt} \tag{2}$$

where r is a number with real and imaginary components. Substituting into (1);

$$m(r^2 e^{rt}) + \gamma(re^{rt}) + k(e^{rt}) = mr^2 + \gamma r + k = 0 \tag{3}$$

using the quadratic formula, we can find the roots to be

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} \tag{4}$$

which means that the homogenous solution, being some linear combination of the solutions given by the roots, is

$$x = C_1 e^{r_+ t} + C_2 e^{r_- t} \tag{5}$$

where C_1 and C_2 are arbitrary constants.

c)

To solve this particular inhomogenous equation, we can predict a solution of the form

$$x(t) = u(\omega)e^{i\omega t} \tag{6}$$

Inserting this into the original equation, we get

$$m(ue^{i\omega t})'' + \gamma(ue^{i\omega t})' + k(ue^{i\omega t}) = e^{i\omega t} \tag{7}$$

and solving for $u(\omega)$, we get

$$u(\omega) = (-m\omega^2 + i\omega\gamma + k)^{-1} \tag{8}$$

3.2

Setting the equilibrium position of each mass to 0, relative to the entire system the spring extensions are x_1, $x_2 - x_1$, and $-x_2$. So, to create a system of equations to equalize all forces,

$$F_{mass1} = F_{spring1} + F_{spring2} \tag{9}$$

$$F_{mass2} = F_{spring2} + F_{spring3} \tag{10}$$

This simplifies to the equations of motion

$$m\ddot{x} = -kx_1 + k(x_2 - x_1) \tag{11}$$

$$m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 \tag{12}$$

or in ODE form,

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \tag{13}$$

$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0 \tag{14}$$

The coefficients form the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{15}$$

which is equivalent to

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = MDM^{-1} \tag{16}$$

so,

$$\frac{k}{m} MDM^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{\partial}{\partial t^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

and

$$\frac{k}{m} D \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{\partial}{\partial t^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{18}$$

because

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = M^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{19}$$

the two modes are given by

$$\frac{3k}{m} y_1 = \ddot{y_1} \tag{20}$$

$$\frac{k}{m} y_2 = \ddot{y_2} \tag{21}$$

showing that one of the modes is a factor of three more than the other.