Nature of Mathematical Modeling - Constrained Optimization

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(16.1) Fit point (x_0, y_0) to closest point on line $y = ax+b$ through maximizing $d^2 = -(x_0-x)^2-(y_0-y)^2 = f(x,y)$ given $y-ax-b=0$. $\Rightarrow g(x,y) = y-ax-b$

First define our Lagrangian:

$$\underline{\nabla f(x,y) - \lambda \nabla g(x,y) = 0}$$

$$\Rightarrow \left.\begin{aligned} 2(x_0 - x) &= -a\lambda \\ 2(y_0 - y) &= \lambda \\ y - ax - b &= 0 \end{aligned}\right\} \Rightarrow \boxed{\lambda = -\left(\frac{2}{a^2+1}\right)(ax_0 - y_0 + b)}$$

Solving this system:

$$\boxed{\begin{aligned} x &= \frac{1}{a^2+1}\left[x_0 + a(y_0-b)\right] \\ y &= \frac{1}{a^2+1}\left(a x_0 + a^2 y_0 + b\right) \end{aligned}}$$

16.2 Given a set of N nodes each measured a quantity x_i:

Find best estimate $\bar{x}$ by ~~minimizing~~: maximizing:

$$\min_{\bar{x}} \sum_{i=1}^{N} (\bar{x} - x_i)^2 \quad (17.29)$$

Each node i communicates only with nodes j in neighborhood $j \in N(i)$.

$$\Rightarrow c_{ij} = \bar{x}_i - \bar{x}_j = 0 \; \forall j \in N(i)$$

(a) Lagrangian?

Two approaches to this problem.

(i) Penalty method
(ii) Augmented Lagrangian Method. (ALM)

Trying ALM:

Recast as a series of unconstrained minimization problems:

(a) Lagrangian $\mathcal{L}^T = \mathcal{L}(\bar{x} - x_i) + \frac{\mu_k}{2} \sum_i c_i'(\vec{x})^2 - \sum_i \lambda_i c_i(\vec{x})$

(b) After each step, update λ: (also μ_k)

$$\lambda_{i+1} \leftarrow \lambda_i - \mu_k c_i(x_k)$$

$$\Updownarrow$$

$$c_i = \frac{(\lambda_i - \lambda_{i+1})}{\mu_k}$$