MAS 864 - Nature of Mathematical Modeling — Christian Wentz

Ch 6 Problems:

2014/3/4

(6.1) (a) First 3 cumulants?

In general: $\log\langle e^{ikx}\rangle = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n$

$$\Rightarrow C_1 \frac{(ik)}{1} = \log\langle e^{ikx}\rangle$$

$$C_1 = \frac{\log\langle e^{ikx}\rangle}{ik}$$

In time domain: $C_n = \frac{\partial^n}{\partial t^n} \underbrace{g(t)}_{\text{moment-generating function}}\Big|_{t=0}$

(b) First 3 cumulants of a Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\bar{x})^2/2\sigma^2}$$

$$\Rightarrow g(t) = \bar{x}t + \frac{\sigma^2 t^2}{2}$$

$$\boxed{C_1 = \frac{\partial}{\partial t} g(t)\Big|_{t=0} = \bar{x}}$$

$$\boxed{C_2 = \frac{\partial^2}{\partial t^2} g(t)\Big|_{t=0} = \sigma^2}$$

$$\boxed{C_3 \ldots C_n = 0}$$

(6.2) $\vec{y}(\vec{x}) = (y_1(x_1, x_2), y_2(x_1, x_2))$ coordinate transformation.

(a) What is area of differential element $dx_1 dx_2$ after mapping to $\vec{y}$?

In one dimension: $p(x) \xrightarrow{y(x)} p(y)$ $\quad p(y)|dy| = p(x)|dx|$

Higher dimensions: $\Rightarrow p(y) = p(x)\left|\frac{dx}{dy}\right|$

multiply original distribution by Jacobian: (absolute val of det of partials)

$$|\det p(y_1, y_2)| = \begin{bmatrix} \left(\frac{\partial x_1}{\partial y_1}\right)^{-1} & \left(\frac{\partial x_1}{\partial y_2}\right)^{-1} \\ \left(\frac{\partial x_2}{\partial y_1}\right)^{-1} & \left(\frac{\partial x_2}{\partial y_2}\right)^{-1} \end{bmatrix} = \underbrace{\left|\left(\frac{\partial x_1}{\partial y_1}\right)^{-1}\left(\frac{\partial x_2}{\partial y_2}\right)^{-1} - \left(\frac{\partial x_2}{\partial y_1}\right)^{-1}\left(\frac{\partial x_1}{\partial y_2}\right)^{-1}\right|}_{A}$$

Transformed volume:

$$dy_1 dy_2 = dx_1 A\, dx_2 A$$

6.2 (b) Let $y_1 = \sqrt{-2\ln(x_1)}\,\sin(x_2)$

$y_2 = \sqrt{-2\ln(x_1)}\,\cos(x_2)$

$p(x_1, x_2)$ = uniform distribution.. what is $p(y_1, y_2)$?

We can explicitly compute $p(y_1, y_2)$ as:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{pmatrix} \frac{-\sin(x_2)}{x_1\sqrt{2\ln(x_1)}} & \sqrt{-2\ln(x_1)}\,\cos(x_2) \\ \frac{-\cos(x_2)}{x_1\sqrt{2\ln(x_1)}} & -\sqrt{-2\ln(x_1)}\,\sin(x_2) \end{pmatrix}$$

$$p(y_1, y_2) = p(x_1, x_2)\det(\mathcal{J}) = \underbrace{[\sin^2(x_2) + \cos^2(x_2)]}_{=1}\, i\, x_1\, p(x_1, x_2)$$

$P(y_1, y_2)$ is also uniform, scaled by $x_1 i$

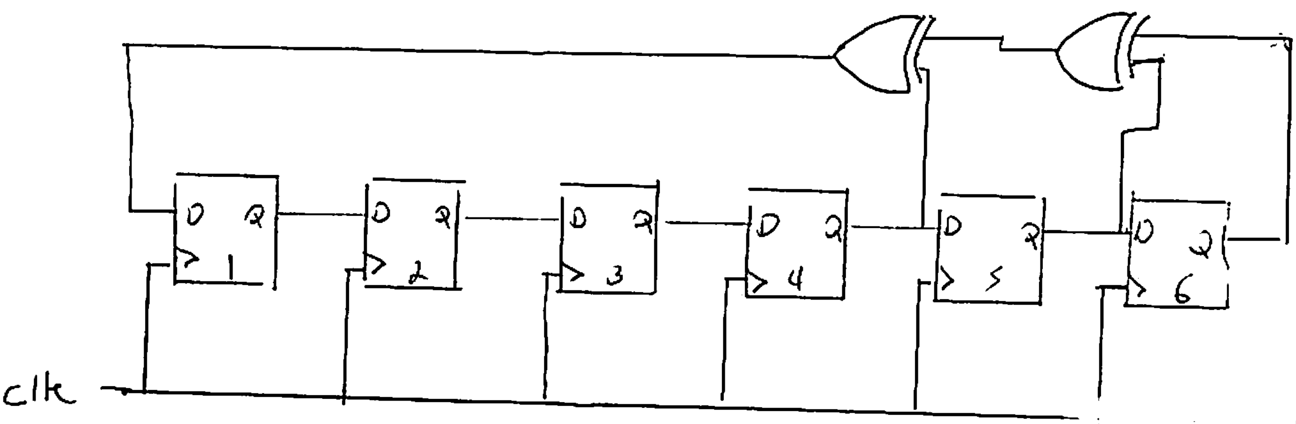
(c) Write random # generator, transform by (b). Numerically evaluate 1st three cumulants

Stay in trig functions:

take two random Uniform #s, only accept those within unit circle, use radius $r = \sqrt{\eta_1^2 + \eta_2^2}$, as x_1, $\eta_1 / r =$

6.2 (c) Addendum:

A conceptual simplification to implementing the LFSR is to cast it explicitly in Boolean logic using XOR gates. To increase period of repetition we can also use a higher order LFSR but take only the lower few bytes (e.g. here I use a 32-bit register but pull only the lower 16). "Further randomness" is achieved here by combining two different LFSRs. Finally, the seed is chosen randomly here but it is important to avoid all zeros, as this may lead to latch-up



$= "x^4 + x^5"$

(not actually an irreducible polynomial)

$2^{(16-1)} = 32$ bit LFSR

We find $c_1 = 0$, $c_2 = 1$, $c_3 = 0$.

$$X_{n} = X_{n-1} + X_{n-4}$$

1: 1 (seed) $\underline{X_{n-1}} \mid \underline{X_{n-4}}$

2: 1 $(1 + 0) \bmod 2 = 1$

3: 1 $(1 + 0) \bmod 2 = 1$

4: 1 $(1 + 0)$

5: 0 $(1 + 1) \bmod 2 = 0$

6: (

7:

8:

9:

10:

Pseudocode:

LFSR = [A B C D E]

$$\text{LFSR out} = \left\{ [A \; B \; C \; D \; E] \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \right. \bmod 2$$

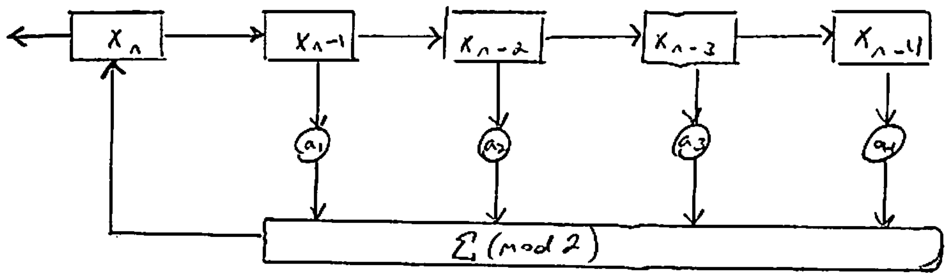
LFSR_shift = circshift(lfsr, [1 1])

~~LFSR_shift~~ LFSR_shift(1,1) = LFSR_out

LFSR = LFSR_shift

(6.3) (a) Bit sequence for 4^{th} order maximal LFSR:

$$X_n = \sum_{i=1}^{M} a_i X_{n-i} \bmod(2)$$



Maximal taps for 4^{th} order use $\underline{X_n = X_{n-1} + X_{n-4}}$ (from table 6.1)

(b) Recurrence time is exponentially dependent on length of register, so goal of 10^{10} years given a 1 GHz clock rate $\Rightarrow \left(10^{10} \text{ years}\right)\left(365 \frac{\text{days}}{\text{year}}\right)\left(\frac{24 \text{ hours}}{\text{day}}\right)\left(\frac{3600 \text{ seconds}}{\text{hour}}\right)\left(\frac{1 \times 10^9 \text{ cycles}}{\text{sec}}\right)$

$$= \left(3.1536 \times 10^7\right)\left(10^{10}\right)\left(10^9\right) = 3.1536 \times 10^{26} \text{ cycles}$$

$$\log\left(3.1536 \times 10^{26}\right) = 26.49 \Rightarrow \boxed{27 \text{ taps}}$$

(6.4) Solve diffusion Eqn using F.T.

"Pi seconds" is a nanocentury ($\sim 10^7 \frac{\text{sec}}{\text{yr}}$)

(a) Assuming initial condition is δ function @ $x=0$:

Time domain $\frac{\partial \rho}{\partial t} = \underbrace{\frac{\langle \delta^2 \rangle}{2\pi}}_{D^{-1}} \frac{\partial^2 \rho}{\partial x^2}$, $\langle \delta^2 \rangle = \int_{-\infty}^{\infty} p_\delta(\delta)\,\delta^2\,d\delta$

$$p(x,0) = \delta(x)$$

$$\delta(t) \Rightarrow 1$$

$$p(x,t) = f(k)\,e^{-\alpha k^2 t}$$

$$S(x,t) = \frac{1}{2\pi}\int \hat{S}(k,t)\,e^{ikx}\,dk = \frac{1}{2\pi}\int e^{-\alpha k^2 t}\,e^{ikx}\,dk = \boxed{\frac{1}{\sqrt{4\pi\alpha t}}\,e^{-\frac{1}{4\alpha t}x^2}}$$

Fundamental solution

(6.4) (b) Variance of diffusion eq'n as $f(t)$?

$$\text{Variance} = \sigma_x^2 = \langle (x-\bar{x})^2 \rangle$$
$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$\text{Second moment} = \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx$$

Our δ function means we start with a Gaussian. Heat loss results in widening of this Gaussian.

Impulse in time domain = $e^{-x^2/2\sigma}$

$$\Rightarrow \sigma_x^2(t) = 2t \cdot D$$
$$(\bar{x} = 0)$$

(c) Diffusion coeff of Brownian motion related to viscosity?

Diffusion coeff $D = \dfrac{k_B T}{k_{drag}}$; $k_{drag} = \dfrac{f_{drag}}{\text{Velocity}}$

$$\text{Viscosity} = \frac{(\text{velocity of flow})(\text{object length})(\text{mass density})}{\text{Reynolds \#}}$$

Brownian motion of small things $\Rightarrow Re \ll 1$:

$$f_{drag} \propto \text{viscosity} \times \text{object length} \times \text{object velocity}$$

so, $\boxed{\text{Diffusion coeff} = \dfrac{k_B T}{k_{drag}} = \dfrac{k_B T \cdot \text{velocity}}{f_{drag}} \propto \dfrac{k_B T}{\text{viscosity} \cdot d}}$

(6.4) (d) Random walk model:

Langevin equation:

random variable

$$\frac{m}{2}\frac{d^2 x^2}{dt^2} - mv^2 = -3\pi\mu a \frac{d(x^2)}{dt} + \eta x$$

parametrize $\Rightarrow$ $m\frac{d^2x}{dt^2} = -\lambda\frac{dx}{dt} + \eta(t)$

Variance $\langle x^2\rangle = Ae^{-6\pi\mu a t/m} + \frac{kT}{3\pi\mu a}t$ $\quad kT = m\langle v^2\rangle$

$$\doteq \frac{kT}{3\pi\mu a}t$$

Pseudocode:

% just approximate as Bernoulli distribution × step size.

(e) Equal probability of ± 1 $\Rightarrow$ Bernoulli distribution

Variance $= 1$, $E\langle x_N^2\rangle = N =$ # of steps × step size

$\sigma(x_n) = \sqrt{N} =$

By normal approximation we'd expect 3σ events to be 99.7% probable