Nature of Mathematical Modeling - ODEs (Ch 7)

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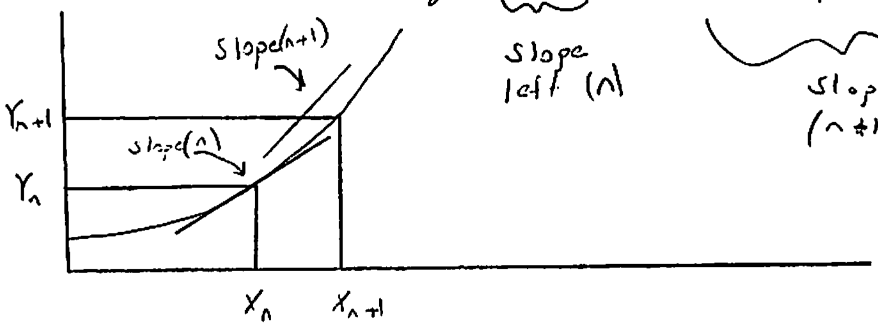
(7.1) 2^{nd} order approximation error of Heun's method: "improved Euler" explicit trapezoidal rule
(average of slopes @ beginning & end of interval)

"Predictor - corrector"

$$y(x+h) = y(x) + \frac{h}{2}\Big\{\underbrace{f(x, y(x))}_{\text{slope left } (n)} + \underbrace{f[x+h, y(x) + h f(x, y(x))]}_{\text{slope } (n+1)}\Big\}$$



Taylor expansion:

$$y(x+h) = y(x) + y'(x)\cdot h + \frac{1}{2}y''(x)h^2 + \frac{1}{6}y'''(x)h^3$$

$$y''(x) = \frac{y'(x+h) - y'(x)}{h} - \frac{1}{2}y'''(x)h$$

$$\Rightarrow y(x+h) = y(x) + y'(x)\cdot h + \frac{1}{2}\left(\frac{y'(x+h) - y'(x)}{h} - \frac{1}{2}y'''(x)h\right)h^2 + \frac{1}{6}y'''h^3$$

$$y(x+h) = y(x) + \frac{1}{2}y'(x)h + \frac{1}{2h}y'(x+h) - \frac{1}{4}y'''(\overset{\tau_2}{x})h^3 + \frac{1}{6}y'''(\overset{\tau_1}{x})h^3$$

$$y'(x+h) = f(x+h, y(x+h))$$

$$= f(x+h, y(x) + h f(x, y(x))) + \frac{1}{2}f_2(\overset{\tau_3}{x}, y(\overset{\tau_3}{x}))y''(x)h^2$$

$$y(x+h) = \underline{y(x)} + \frac{h}{2}y'(x) + \frac{h}{2}(f(x+h, y(x) + h y'(x)))$$

$$+ \boxed{\left[\frac{1}{6}y'''(\tau_1) - \frac{1}{4}y'''(\tau_2) + \frac{1}{4}f_2(\tau_3, y(\tau_3))y''(\tau_3)\right]h^3}$$

Error

$\therefore$ error is $\mathcal{O}(h^3)$.

(7.2) S.H.O., $y'' + y = 0$; $y(0) = 1$; $y'(0) = 0$

Find $y(t)$ for $t = 0$ to 100π for Euler & fixed-step 4^{th}-order Runge-Kutta.

Find average local error, error in final value & slope, dependence on step size.

		error
4^{th}-order Runge-Kutta:	$y_i(x+h) = y_i(x) + \frac{k_{1,i}}{6} + \frac{k_{2,i}}{3} + \frac{k_{3,i}}{3} + \frac{k_{4,i}}{6}$	$(+\,\mathcal{O}(h^5))$
Euler:	$y(x+h) = y(x) + h \cdot f(x, y(x))$	(error $\mathcal{O}(h^2)$)