MAS.864: Nature of Mathematical Modeling - (8) - PDEs. 2014/3/18

[Hand Calculations - see repo for code] Christian Wentz

(8.1) 1D wave eq'n:

$$\frac{\partial^2 u}{\partial t^2} = v\frac{\partial^2 u}{\partial x^2}$$

(a) Finite difference approx: (centered)

Generally: $f''(x) \approx \frac{\delta_h^2 f(x)}{(\Delta x)^2} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$

So for wave eq'n:

$$\boxed{\frac{1}{v}\frac{(\Delta x)^2}{(\Delta t)^2}\, f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t) = f(x,t+\Delta x) - 2f(x,t) + f(x,t-\Delta x)}$$

(b) This is accurate to 1st order in both time & space

(c & d) Von Neumann stability criterion gives stability amplitudes of:

(parameterize to isolate Δt): desire $|A(k)| < 1$

We know that solutions will be sinusoidal, so:

Guess form of $f(x,t) = \cos(kx \pm \omega t)$

$$\Rightarrow u_m^n = A(k)^n e^{jkm\Delta x} e^{j\omega n\Delta t}$$

In F.D.:

$$\frac{1}{v}\frac{(\Delta x)^2}{(\Delta t)^2}\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n\right] = u_m^{n+1} - 2u_m^n + u_m^{n-1}$$

$$\Rightarrow u_m^{n+1} = \frac{1}{v}\left(\frac{\Delta x}{\Delta t}\right)^2\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) + 2u_m^n - u_m^{n-1}$$

Putting together: modes are of form $A(k)^n e^{jkm\Delta k}$

$$A_m^{n+1} e^{jmk\Delta x} e^{j\omega(n+1)\Delta t} =$$

n = time
m = space

$$\underbrace{\frac{1}{v}\left(\frac{\Delta x}{\Delta t}\right)^{-2}}_{=s}\left[A_m^n e^{j(m+1)k\Delta x} e^{j\omega n\Delta t} - 2A_m^n e^{jmk\Delta x} e^{j\omega n\Delta t} + A_{m-1}^n e^{j(m-1)k\Delta x} e^{j\omega n\Delta t}\right]$$

$$+ 2A_m^n e^{jmk\Delta x} e^{j\omega n\Delta t} - A_m^{n-1} e^{jmk\Delta x} e^{j\omega(n-1)\Delta t}$$

Rearranging:

$$A_m^{n+1} = s\left(A_{m+1}^n e^{(\dots)} + A_{m-1}^n e^{(\dots)}\right) + 2(1-s)A_m^n e^{(\dots)} - A_m^{n-1}$$

$$\Rightarrow A_m^n + \frac{1}{A_m^n} = s\left(e^{jk\Delta x} + e^{-jk\Delta x}\right) + 2(1-s)$$

$$= \underbrace{2 + 2s(\cos k\Delta x - 1)}_{=p}$$

Parameterize:

$$A + \frac{1}{A} = p \Rightarrow A^2 - Ap + 1 = 0$$

$$A = \frac{p \pm \sqrt{p^2 - 4}}{2} \Rightarrow |p| \leq 2.$$

Plugging back in:

$$-2 \leq 2s(\cos kx - 1) \Rightarrow \cos kx \leq 1$$

$$\Rightarrow s \leq 1 \Rightarrow \boxed{\frac{1}{v}\left(\frac{\Delta t}{\Delta x}\right)^2 \leq 1}$$

part(d) solution: condition on Δt, Δx, v

(e). Yes, the particular solution is dependent on k.

(f) "Numerically solve wave eq'n for u=0 (i.c.) with one non-zero node."

[see code]

(g) "Replacing wave eq'n with:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2} \quad ; \quad \text{assuming} \quad u(x,t) = A e^{i(kx - \omega t)}$$

Find relationship between k, ω;
simplify for small γ."

This " $\gamma \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2}$ " term is the numerical dissipation that depends on discretization. For small γ, this returns to normal wave equation.

Numerically solve wave equation for $u=0$ initial condition, with one non-zero node, evaluate stability criterion.

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

So at $t=0$, we know $F(x,0)$ (zeroes except for $F(250)=1$)

advancing to $t=dt$, we use an update rule starting at $x = x_{0-1}$, $x=x_0$, $x=x_{0+1}$

to compute $F(x_0)$ at $t=dt$.

repeat until done.

So what is update rule?

1st order central difference to start:

$$\frac{\partial}{\partial t}(x) = \frac{F(x+\Delta x) - F(x-\Delta x)}{2\Delta x}$$

or, on a grid:

$$f(x_n) = \frac{[u(x_{n+1}) - u(x_{n-1})]}{\Delta x}$$

$$\frac{\partial^2 F(x)}{\partial x^2} \cong \frac{\partial}{\partial x}\left[\frac{F(x+\Delta x/2) - F(x-\Delta x/2)}{\Delta x}\right]$$

$$\cong \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{(\Delta x)^2}$$

$r = c\Delta t/\Delta x$:

$$\Rightarrow r^2(u^n_{j+1} - 2u^n_j + u^n_{j-1}) = u^{n+1}_j - 2u^n_j + u^{n-1}_j$$

$$\Rightarrow \boxed{u^{n+1}_j = r^2(u^n_{j+1} - 2u^n_j + u^n_{j-1}) + 2u^n_j - u^{n-1}_j}$$

For $\boxed{u^1_j: \quad u^{n+1}_j = -\frac{1}{2} r^2(u^n_{j+1} - 2u^n_j + u^n_{j-1}) + u^0_j}$

8.2 (a) Courant condition for diffusion eq'n:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \qquad D = \Delta x = 1$$

$$\frac{2D\Delta t}{(\Delta x)^2} \leq 1 \Rightarrow \Delta t \leq \left(\frac{2D}{(\Delta x)^2}\right)^{-1}$$

$$\boxed{\Delta t \leq 2^{-1}}$$