MAS 864 Nature of Mathematical Modeling (9) - Finite Elements.

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(9.1) Damped wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2}$$

Solution domain to be interval $[0,1]$

(a) Find approximating system of equations using Galerkin:

$$u(\vec{x},t) \cong \sum_i a_i(t)\,\phi_i(\vec{x})$$

Galerkin. Residual definition.

$$\int R(\vec{x},t)\, w_i(\vec{x},t)\, d\vec{x} = 0 . \qquad \left(\overset{\text{Residual}}{R(\vec{x},t)} = D\big[\overset{\text{approximation}}{\tilde{u}(\vec{x},t)}\big] - \overset{\text{actual}}{f(\vec{x},t)} \right)$$

Choose weighting function s.t. $w_i(\vec{x}) = \frac{\partial u}{\partial a_i} = \phi_i(\vec{x})$
(Bubnov - Galerkin)

$$\Rightarrow \int \left[\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)\right] \phi_j\, dx = 0$$

$$\sum_i \left[\int_0^1 \frac{d^2 a_i(t)}{dt^2}\big(\phi_i(x)\,\phi_j(x)\big)\, dx - \frac{d^2\phi_i}{dx^2}\left(v^2 + \gamma \frac{d a_i(t)}{dt}\right) \phi_j(x)\, dx \right]$$

Rearranging:

$$\boxed{\begin{aligned} &\sum_i \Bigg[\bigg(\frac{d^2 a_i}{dt^2} \underbrace{\int_0^1 \phi_i(x)\,\phi_j(x)}_{\text{"}A_{ij}\text{"}} \bigg) - \frac{d a_i}{dt}\gamma \underbrace{\int_0^1 \frac{d^2\phi_i}{dx^2}\phi_j(x)\,dx}_{\text{"}B_{ij}\text{"}} - v^2 a_i \underbrace{\int_0^1 \frac{d^2}{dx^2}\phi_i\,\phi_j(x)\,dx}_{\text{"}B_{ij}\text{"}} \\ &\text{NoMM notation:} \\ &= \sum_i A \cdot \frac{d^2}{dt^2} a_i - B \cdot \frac{d}{dt} a_i - C a_i \end{aligned}}$$

(b) Matrix coeffs for linear hat basis functions of fixed size h:

$$\phi_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \le x < x_i \quad = \dfrac{x - x_i - h}{h} \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i \le x < x_{i+1} \quad = \dfrac{x_i + h - x}{h} \end{cases}$$

$$A_{ij} = \int \phi_i \phi_j \, dx = \int_{x_{i-h}}^{x_i} \left(\frac{x - (x_i - h)}{h} \right)^2 dx + \int_{x_i}^{x_{i+h}} \left(\frac{(x_i + h) - x}{h} \right)^2 dx$$

$$= \int_{x_{i-h}}^{x_i} \frac{x^2 - 2x(x_i - h) + (x_i - h)^2}{h^2} \, d(x_{i-h}) + \int_{x_i}^{x_{i+h}} \frac{x_i^2 + 2x_i h - 2x x_i - 2xh + h^2 + x^2}{h^2} \, dx$$

$$= \frac{1}{h^2} \left[x^2 (x_i - h) - x (x_i - h)^2 + \frac{(x_i - h)^3}{3} \right]_{x_{i-h}}^{x_i}$$

$$+ \frac{1}{h^2} \left[\frac{x_i^3}{3} + x_i^2 h + x_i^2 x - x h x_i^2 + h^2 x_i + x^2 x_i \right]_{x_i}^{x_{i+h}}$$

$$= \frac{1}{h^2} \Big(x^2 (-h) - x \left((x_i - h)^2 - x_i^2 \right) + \frac{(x_i - h)^3 - x_i^3}{3}$$

$$+ \frac{x_i^3 - (x_i + h)^3}{3} + h \left(x_i^2 - (x_i + h)^2 \right) - x \left(x_i^2 - (x_i + h)^2 \right)$$

$$- x h \left(x_i^2 - (x_i + h)^2 \right) + h^2 \left(x_i^2 - (x_i + h)^2 \right)$$

$$+ x^2 \left(x_i - (x_i + h) \right) \Big)$$

$$= \frac{1}{h^2}\Big[-hx^2 - x\big(\cancel{x_i^2} - 2hx_i + h^2 - \cancel{x_i^2}\big) + \frac{1}{3}\big(\cancel{x_i^3} - \cancel{2x_i^2h} + \cancel{x_ih^2} - \cancel{hx_i^2} - \cancel{2x_ih^2} - \cancel{h^3}\big) \cancel{-\frac{x_i^3}{3}} \Big]$$

$$+ \frac{1}{h^2}\Big[\frac{x_i^3}{3} + \frac{1}{3}\big(x_i^3 + \cancel{2x_ih} \cancel{x_ih^2} + \cancel{hx_i^2} + \cancel{2x_ih^2} + \cancel{h^3}\big)$$

$$+ hx_i^2 - x_i^2h + 2x_ih^2 + h^3 - x\big(x_i^2 - x_i^2 + 2x_ih + h^2\big)$$

$$- xh\big(x_i^2 - (x_i^2 + 2x_ih + h^2)\big)$$

$$+ h^2x_i^2 - h^2(x_i^2 + 2x_ih + h^2)$$

$$+ x^2x_i - x^2(x_i^2 + 2x_ih + h^2) \Big]$$

$$\boxed{A_{ij}^{1} = \frac{-2h}{3}}$$

$$B_{ij} = \int \frac{d^2\phi_i}{dx^2}\,\phi_j(x)\,dx$$

$$= \int_{x_{i-h}}^{x_i} \frac{1}{h^2}\,dx + \int_{x_i}^{x_i+h} \frac{1}{h^2}\,dx$$

$$\boxed{B_{ij} = \frac{2}{h}}$$

(c) see code (matlab)

9.2 see code (matlab)