(9.1) $$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2}$$

solution domain $[0, 1]$

A. Galerkin method to find approximating system of diff eqs

$$R(\vec{x}, t) = D[\hat{u}(\vec{x}, t)] - f(\vec{x}, t)$$

$$u(\vec{x}, t) \approx \sum_i a_i(t)\, \varphi_i(\vec{x})$$

$$\int R(\vec{x})\, w_i(\vec{x})\, d\vec{x} = 0 \qquad w_i(\vec{x}) = \varphi_i(\vec{x})$$

$$\int \left(\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2} \right) \varphi_j\, dx = 0$$

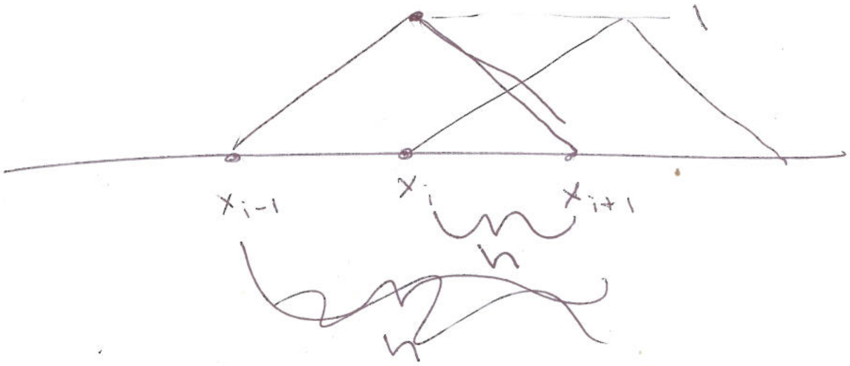
$$\sum_i \int \left(\frac{\partial^2 a_i}{\partial t^2} \varphi_i - v^2 a_i \frac{\partial^2 \varphi_i}{\partial x^2} - \gamma \frac{\partial a_i}{\partial t} \frac{\partial^2 \varphi_i}{\partial x^2} \right) \varphi_j\, dx = 0$$

$$= A \frac{\partial^2 a_i}{\partial t^2} - v^2 B a_i - \gamma B \frac{\partial a_i}{\partial t} = 0$$

$$A_{ij} = \int \varphi_i \varphi_j\, dx$$

$$B_{ij} = \int \varphi_j \frac{\partial^2 \varphi_i}{\partial x^2}\, dx$$

B. Eval matrix coefficients for linear hat basis function, using elements of fixed size of h



$$\varphi_i = \begin{cases} (x - x_{i-1})/(x_i - x_{i-1}) & x_{i-1} \le x \le x_i \\ (x_{i+1} - x)/(x_{i+1} - x_i) & x_i \le x \le x_{i+1} \\ 0 & x < x_{i-1} \,\|\, x \ge x_{i+1} \end{cases}$$

if $i = j$ $\quad A_{ij} = \int \varphi_i^2 \, dx$

$$= \int_{x_{i-1}}^{x_{i+1}}$$

$$= \int_{-h}^{0} \left(\frac{(x - h)}{h} \right)^2 dx + \int_0^h \left(\frac{(h - x)}{h} \right)^2 dx$$

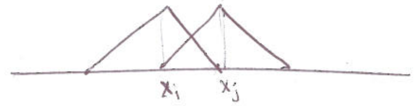
$$= 2 \int_0^h \left(\frac{h - x}{h} \right)^2 dx$$

$$= 2 \int_0^h (1 - x/h)^2 \, dx$$ ← help from wolfram alpha

$$= 2 \left(\frac{x^3}{3h^2} - \frac{x^2}{h} + x \right) \Big|_0^h$$

$$\boxed{A_{ij} = 2h/3}$$

if $i = j \pm 1$:



$$\varphi_j = (x - x_i)/(x_j - x_i)$$

$$\varphi_i = (x_j - x)/(x_j - x_i)$$

$x_i \le x \le x_j$

$$A_{ij} = \int_{x_i}^{x_j} \frac{(x-x_i)}{(x_j-x_i)} \frac{(x_j-x)}{(x_j-x_i)} \, dx$$

let $x_i = 0$
$x_j = h$

$$= \frac{1}{h^2}\int_0^h (x-0)(h-x)\,dx$$

$$= 1/h^2 \int_0^h (hx - x^2)\,dx$$

$$= 1/h^2 \left(hx^2/2 - x^3/3\right)\Big|_0^h$$

$$A_{ij} = h/6$$

all other $A_{ij} = 0$

* except boundaries:

$$A_{ij} \,\big|\, i = j = 0 \;||\; i = j = n$$

$$= (h/3)\,1/2 = h/6$$

for B_{ij}:

$$\frac{\partial^2 \varphi_i}{\partial x^2} = 0 \text{ for all } x$$

so all $B_{ij} = 0$

c. Find matrix coefficients for Hermite polynomial interpolation basis functions

elements of fixed size h

1D cubic polynomial

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\begin{bmatrix} u_0 \\ \dot{u}_0 \\ u_h \\ \dot{u}_h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h & h^2 & h^3 \\ 0 & 1 & 2h & 3h^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

invert

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h^2 & -2/h^3 & 1/h^2 \end{bmatrix}$$

$1/4 (1-\xi)^2 (2+\xi)$

$1/4 (1+\xi)^2 (2-\xi)$

$a_0 = u_0$

$a_1 = \dot{u}_0$

$a_2 = u_0 + h\dot{u}_0 + h^2 u_h + h^3 \dot{u}_h$

$a_3 = \dot{u}_0 + 2h u_h + 3h^2 \dot{u}_h$

derivs wrt x

(9.2) Hermite polynomial shape functions to numerically solve the beam bending equation.

fix disp + slope @ one end

$$V = \int_0^L \left(\tfrac{1}{2} EI \left(\frac{\partial^2 u}{\partial x^2} \right)^2 - u(x) f(x) \right) dx$$

equilibrium given by $\delta V = 0$

$$u(x,t) = \sum_i a_i(t)\, \varphi_i(x)$$

$$V = \int_0^L \left(\tfrac{1}{2} EI \left(\sum a_i(t) \frac{\partial^2 \varphi_i}{\partial x^2} \right)^2 - \sum a_i(t)\, \varphi_i(x) f(x) \right) dx$$

$$\frac{\partial V}{\partial a_j} = 0$$

$$0 = \frac{\partial}{\partial a_j} \Big(\qquad \Big)$$

$$\frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial^2 \varphi_j}{\partial x^2}$$

$$= \int_0^L \left(\tfrac{1}{2} EI \sum a_i \cancel{\left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2} - \varphi_i f(x) \right) dx$$

$$= \sum a_i \int_0^L \tfrac{1}{2} EI \cancel{\left(\frac{\partial^2 \varphi}{\partial x^2} \right)^2} dx - \int_0^L \varphi_j f(x)\, dx$$

$$A_{ij} = \int_0^L EI \frac{\partial^2 \varphi_i}{\partial x^2} \frac{\partial^2 \varphi_j}{\partial x^2} dx$$

$$\vec{B}_j = \int_0^L \varphi_j(x) f(x)\, dx$$

$$A \cdot \vec{a} = \vec{B}$$