MAS.864 Transforms

12.1)

$$X_f = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{2\pi i f n/N} x_n$$

$$= \sum_{n=0}^{N-1} M_{fn} x_n$$

$$X = M \cdot \vec{x}$$

$$x_n = \frac{1}{\sqrt{N}} \sum_{f=0}^{N-1} e^{-2\pi i f n/N} X_f$$

$$= \frac{1}{\sqrt{N}} \sum_{f=0}^{N-1} \left(e^{2\pi i f n/N}\right)^* X_f$$

$$= \frac{1}{\sqrt{N}} \sum_{f=0}^{N-1} (M_{fn})^* X_f$$

$M_{fn} = M_{nf}$ since $e^{2\pi i ab/N} = e^{2\pi i ba/N}$

hence $M = M^T$

$$\vec{x} = (M)^* X$$

$$= (M^T)^* X$$

$$\vec{x} = M^\dagger M \vec{x}$$

Hence M and the DFT are unitary

12.2

1s in 5th and 30th place of $\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ w_0 \\ w_1 \\ \vdots \end{bmatrix}$

$\downarrow$

1s in 9th and 59th place of $\begin{bmatrix} x_0 \\ w_0 \\ x_1 \\ w_1 \\ \vdots \end{bmatrix}$

9th col of inverse matrix =

$$\begin{array}{c} \text{row} \\ \\ 5 \\ 6 \\ 7 \\ 8 \\ \\ \\ \end{array} \begin{bmatrix} \vdots \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \\ \vdots \end{bmatrix} \quad \lfloor 9/2 \rfloor = 4$$

59th col of inverse matrix =

$$\begin{array}{c} \\ \\ 30 \\ 31 \\ 32 \\ 33 \\ \\ \end{array} \begin{bmatrix} \vdots \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ 0 \end{bmatrix} \quad \left\lfloor \frac{59}{2} \right\rfloor = 29$$

inverse wavelet transform =

$$\begin{array}{c} \\ \\ 5 \\ 6 \\ 7 \\ 8 \\ \\ \\ 30 \\ 31 \\ 32 \\ 33 \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \\ \vdots \\ 0 \\ c_0 \\ c_1 \\ c_2 \\ c_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$

$c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$

$c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$

$c_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$

12.3

a.)

$$f_1(x) = f_2(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}}$$

$$f_3(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{-\frac{(x-y)^2}{2}} dy$$

$$= \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} \quad \text{Variance} = 2$$

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\mathrm{Cov}(X_3, X_1) = E[(X_1)(X_3)]$$

$$= E[(X_1)(X_1 + X_2)]$$

$$= E[(X_1)^2 + (X_1 X_2)^2]$$

$$= E[(X_1)^2] + 0 = 1$$

$$\mathrm{Cov}(X_3, X_2) = \mathrm{Cov}(X_3, X_1) = 1$$

b.)

$$\left| \begin{bmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & -1 \\ -1 & -1 & \lambda-2 \end{bmatrix} \right| =$$

$$(\lambda-1)((\lambda-1)(\lambda-2) - 1) \quad -1(\lambda-1) = 0$$

$$(\lambda-1)^2(\lambda-2) - 2(\lambda-1) = 0$$

$$\cancel{2}\lambda^3 - 4\lambda^2 + 5\lambda - 2\lambda \cancel{+2} = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$$(\lambda-3)(\lambda-1)\lambda = 0$$

$$\lambda = 3, 1, 0$$