12.2

a) $$d(x,z) = -De^{ikx^2/2z}\left(\frac{xD}{z\lambda}\right)$$

where $(u) = \frac{\sin \pi u}{\pi u}$, D is the lens aperture width, λ is the wavelength, and $k = \frac{2\pi}{\lambda}$. This the point spread function (PSF) is showing how light from a point spreads out.

The intensity is proportional to the square of the amplitude:

$$|d(x,z)|^2 \propto D^{22}\left(\frac{xD}{z\lambda}\right)$$

At $x=0$, $(0) = 1$, so the maximum intensity is $\propto D^2$

A 3 dB drop means the intensity is halved:

$$\frac{|d(x,z)|^2}{|d(0,z)|^2} = 2\left(\frac{xD}{z\lambda}\right) = 0.5$$

Let $u = \frac{xD}{z\lambda}$, so we need:

$$(u) = \frac{\sin(\pi u)}{\pi u} = \sqrt{0.5} \approx 0.707$$

Solve:

$$\sin(\pi u) = 0.707\pi u$$

$\pi u \approx 1.391$, so $u \approx 0.4426$, Thus:

$$\frac{xD}{z\lambda} = 0.4426 \quad x = 0.4426\frac{z\lambda}{D}$$

The resolving power is the distance between $-x$ and $+x$:

$$\delta x = 2 \times 0.4426\frac{z\lambda}{D} \approx 0.8852\frac{z\lambda}{D}$$

Rayleigh's criterion says two points are resolved when the maximum of one PSF is at the first minimum of the other. The function's first zero is at:

$$\pi\frac{xD}{z\lambda} = \pi\frac{xD}{z\lambda} = 1 \quad x = \frac{z\lambda}{D}$$

So : $\delta x = \frac{z\lambda}{D}$

b) Ultrasound has frequency $f = 100\,kHz = 100{,}000\,Hz$, and speed in air $v \simeq 350\,m/s$ The wavelength is :

$$\lambda = \frac{v}{f} = \frac{350}{100{,}000} = 0.0035\,m = 3.5\,mm$$

Distance $z = 1m$, resolution $\delta x = 1cm, = 0.01m$

Using $\delta x = \frac{z\lambda}{D}$; rearrange for D:

$$D = \frac{z\lambda}{\delta x} = \frac{1 \times 0.0035}{0.01}, = \frac{0.0035}{0.01} = 0.35m$$

c) Radar frequency $f = 10GHz = 10 \times 10^9\,Hz$, speed of light $c = 3 \times 10^8\,m/s$:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 0.03m \quad 3cm$$

Distance $z = 200km, = 200{,}000\,m = 2 \times 10^5\,m$, resolution $\delta x = 1cm = 0.01m$

$$D = \frac{z\lambda}{\delta x} = \frac{2 \times 10^5 \times 0.03}{0.01} = \frac{2 \times 0.03 \times 10^5}{0.01} = \frac{0.06 \times 10^5}{0.01} =$$

$$6 \times 10^3\,m = 6000m = 6km$$

The angle subtended by the aperture is the angle it spans from the Earth's surface at distance z. For a width D, the angle in radians is :

$$\phi \simeq \frac{D}{z} = \frac{6000}{2 \times 10^5} = \frac{6 \times 10^3}{2 \times 10^5} = 0.03 \text{ radians}$$

To make this clearer, convert to degrees:

$$\phi \simeq 0.03 \times \frac{180}{\pi} \simeq 0.03 \times 57.3 \simeq 1.72^\circ$$

$\boxed{12.3}$ The distance from the transceiver at $(x', 0)$ to the target at $(0, z)$ is:

$$r = \sqrt{(x'-0)^2 + (0-z)^2} = \sqrt{x'^2 + z^2}$$

The wave travels to the target and back, so the total path is $2r$. The phase of the received signal is:

$$\phi = k \cdot 2r = 2kr, \quad \text{where } k = \frac{2\pi}{\lambda}$$

So, the signal received at x' is proportional to:

$$e^{i2kr} = e^{i2k\sqrt{x'^2 + z^2}}$$

The problem says "paraxial," meaning $x' \ll z$. We approximate the distance:

$$r = \sqrt{x'^2 + z^2} = z\sqrt{1 + \frac{x'^2}{z^2}} \approx z\left(1 + \frac{x'^2}{2z^2}\right) = z + \frac{x'^2}{2z}$$

So: $$2kr \approx 2k\left(z + \frac{x'^2}{2z}\right) = 2kz + \frac{kx'^2}{z}$$

The signal's phase is:

$$e^{i2kr} \approx e^{i2kz} e^{ikx'^2/z}$$

To focus on $(0, z)$, the matched filter applies a phase shift opposite to the signal's phase to align all signals:

$$g(x') = \left(e^{i2kr}\right)^* = e^{-i2kr} \approx e^{-i2kz} e^{-ikx'^2/z}$$

The e^{-i2kz} term is a constant phase (since z is fixed) and doesn't effect the focusing, so we can use:

$$g(x') = e^{-ikx'^2/z}$$

$\boxed{12.4}$ $\omega = \gamma B$

where γ is gyromagnetic ratio (different for nuclei and electrons) and B is magnetic field strength. The frequency in hertz is :

$$f = \frac{\omega}{2\pi} = \frac{\gamma B}{2\pi}$$

Here, $B = 1T$ (tesla)

For a particle with charge q and mass m :

$$\gamma = \frac{q}{2m}$$

In SI units, q is in coulombs, m in kilograms, so γ is in rad/s/T.

For an electron, charge $q = -e = -1.602 \times 10^{-19} C$, mass $m_e = 9.109 \times 10^{-31}$ kg. The magnitude of the charge is e, so :

$$\gamma_e \approx \frac{e}{2m_e} = \frac{1.602 \times 10^{-19}}{2 \times 9.109 \times 10^{-31}} = \frac{1.602}{1.8218} \times 10^{11} \approx 0.879 \times 10^{11} \text{ rad/s/T}$$

Actually, electrons have a g-factor $g \approx 2$, so:

$$\gamma_e \approx g \cdot \frac{e}{2m_e} \approx 2 \times 0.879 \times 10^{11} \approx 1.76 \times 10^{11} \text{ rad/s/T}$$

Frequency :

$$f_e = \frac{\gamma_e B}{2\pi} = \frac{1.76 \times 10^{11} \times 1}{2 \times 3.1416} \approx \frac{1.76 \times 10^{11}}{6.2832} \approx 2.8 \times 10^{10} \text{ Hz} = 28 \text{ GHz}$$

For a proton (common in NMR), charge $q = +e = 1.602 \times 10^{-19}$ C, mass $m_p \approx 1.673 \times 10^{-27}$ kg:

$$\gamma_p = \frac{e}{2m_p} = \frac{1.602 \times 10^{-19}}{2 \times 1.673 \times 10^{-27}} = \frac{1.602}{3.346} \times 10^{8} \approx 0.4788 \times 10^{8} \text{ rad/s/T}$$

Protons have a g-factor $g \approx 5.585$, so:

$$\gamma_p = 2.7927 \times \frac{e}{2m_p} \approx 2.7927 \times 0.4788 \times 10^{8} \approx 1.337 \times 10^{8} \text{ rad/s/T}$$

More precisely, $\gamma_p \approx 2.675 \times 10^{8}$ rad/s/T. Frequency:

$$f_p = \frac{2.675 \times 10^{8} \times 1}{2 \times 3.1416} \approx \frac{2.675 \times 10^{8}}{6.2832} \approx 4.26 \times 10^{7} \text{ Hz} = 42.6 \text{ MHz}$$

$\boxed{12.5}$ a) A charge q is at $(0,0,h)$ and the conducting plane is at $z=0$. The plane is grounded, so its potential is zero.

To make the potential zero at $z=0$, place an image charge $-q$ at $(0,0,h)$. The potential for $z \geq 0$:

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-h)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+h)^2}}\right)$$

At $z=0$:

$$V(x,y,0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+h^2}} - \frac{1}{\sqrt{x^2+y^2+h^2}}\right) = 0$$

The surface charge density is $\sigma = \epsilon_0 E_z$ at $z=0$, where $E_z = -\frac{\partial V}{\partial z}$:

$$\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\left(-\frac{z-h}{(x^2+y^2+(z-h)^2)^{3/2}} + \frac{z+h}{(x^2+y^2+(z+h)^2)^{3/2}}\right)$$

At $z=0$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{q}{4\pi\epsilon_0}\left(-\frac{-h}{(x^2+y^2+h^2)^{3/2}} + \frac{h}{(x^2+y^2+h^2)^{3/2}}\right) =$$

$$\frac{q}{4\pi\epsilon_0} \cdot \frac{2h}{(x^2+y^2+h^2)^{3/2}}$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{qh}{2\pi\epsilon_0 (x^2+y^2+h^2)^{3/2}}$$

$$\sigma = \epsilon_0 E_z = -\frac{qh}{2\pi(x^2+y^2+h^2)^{3/2}}$$

$$\sigma(x,y) = -\frac{qh}{2\pi(x^2+y^2+h^2)^{3/2}}$$

Part (b): The plane is divided into squares of side a.

The electrode at indices (m,n) covers:

$$x \in \left[\left(m-\frac{1}{2}\right)a, \left(m+\frac{1}{2}\right)a\right], \; y \in \left[\left(n-\frac{1}{2}\right)a, \left(n+\frac{1}{2}\right)a\right]$$

The charge on electrode (m,n) is:

$$Q_{m,n} = \int_{(m-\frac{1}{2})a}^{(m+\frac{1}{2})a} \int_{(n-\frac{1}{2})a}^{(n+\frac{1}{2})a} \sigma(x,y)\, dx\, dy$$

$$Q_{m,n} = -\frac{qh}{2\pi}\int_{(m-\frac{1}{2})a}^{(m+\frac{1}{2})a}\int_{(n-\frac{1}{2})a}^{(n+\frac{1}{2})a}\frac{dx\,dy}{(x^2+y^2+h^2)^{3/2}}$$

This integral is complex. lets try a substitution to simplify, using $u = \frac{x}{h}$, $v = \frac{y}{h}$, so $dx\,dy = h^2 du\,dv$ and:

$$x^2+y^2+h^2 = h^2(u^2+v^2+1)$$

The limits change: for x from $(m-\frac{1}{2})a$ to $(m+\frac{1}{2})a$, u from $\frac{(m-\frac{1}{2})a}{h}$ to $\frac{(m+\frac{1}{2})a}{h}$. So:

$$Q_{m,n} = \frac{qh}{2\pi}\cdot h^2\int_{\frac{(m-1/2)a}{h}}^{\frac{(m+\frac{1}{2})a}{h}}\int_{\frac{(n-\frac{1}{2})a}{h}}^{\frac{(n+1/2)a}{h}}\frac{du\,dv}{h^3(u^2+v^2+1)^{3/2}} =$$

$$-\frac{q}{2\pi}\iint\frac{du\,dv}{(u^2+v^2+1)^{3/2}}$$

$$Q_{m,n} = -\frac{qh}{2\pi}\int_{(m-\frac{1}{2})a}^{(m+\frac{1}{2})a}\int_{(n-\frac{1}{2})a}^{(n+\frac{1}{2})a}\frac{dx\,dy}{(x^2+y^2+h^2)^{3/2}}$$

Part C $q = 1C$, $h = 1m$, $a = 1m$, and consider a 5×5 grid ($m,n = -2$ to 2).

For $m=0$, $n=0$, the electrode is from $x = -0.5$ to 0.5, $y = -0.5$ to 0.5.

The integral is:

$$Q_{0,0} = -\frac{1\cdot1}{2\pi}\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\frac{dx\,dy}{(x^2+y^2+1)^{3/2}}$$

To approximate, assume the integrand is nearly constant over the small square. At the center $(0,0)$:

$$\frac{1}{(0^2+0^2+1)^{3/2}} = 1$$

The area is $1 \times 1 = 1m^2$, but the integrand decreases away from the center. Try the average radius:

$$\text{Average } (x^2+y^2)^{1/2} \approx \frac{0.5}{\sqrt{2}} \approx 0.3536, \; x^2+y^2 \approx 0.125$$

$$(0.125+1)^{3/2} \approx 1.125^{3/2} \approx 1.19$$

$$Q_{0,0} \approx -\frac{1}{2\pi} \cdot \frac{1}{1.19} \cdot 1 \approx -\frac{0.84}{6.2832} \approx -0.134C$$

$\boxed{\text{Part d}}$ Assume possible source charges q_k at heights $z_k = k\Delta z$ above $(0,0)$. Each contributes to $Q_{m,n}$.

$$Q_{m,n} = \sum_k q_k \left(-\frac{z_k}{2\pi} \iint \frac{dx\,dy}{(x^2+y^2+z^2_k)^{3/2}} \right)$$

Write as $Q = Aq$, where $A_{m,n,k}$ is the integral for height z_k.

$$\text{Minimize} \quad \chi^2 = \sum_{m,n} \left(Q_{m,n} - \sum_k A_{m,n,k} q_k \right)^2 + \lambda \sum_k q^2_k$$

The solution is:

$$q = (A^T A + \lambda I)^{-1} A^T Q$$