S.1 a):

$$P_n(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\log P_n(x) = \log n! - \log(n-x)! - \log x! + x\log p + (n-x)\log(1-p)$$

$$\simeq n\log n - (n-x)\underbrace{\log(n-x)}_{\simeq \log n - x/n} - \log x!$$

$$+ x\log p + (n-x)\underbrace{\log(1-p)}_{\simeq -p}$$

$$\simeq x\log(np) - (np) - \log x!$$

$$\Rightarrow P_n(x) \simeq \frac{(np)^x e^{-np}}{x!}$$

S.1 b) $\langle x(x-1)\cdots(x-m+1)\rangle$

$$= \sum_{x=0}^{\infty} \frac{e^{-N} N^x}{x!} \underbrace{x(x-1)\cdots(x-m+1)}_{0 \text{ for } x<m}$$

$$= \sum_{x=m}^{\infty} \frac{e^{-N} N^x}{x!} x(x-1)\cdots(x-m+1)$$

$$= \sum_{x=m}^{\infty} \frac{e^{-N} N^x}{(x-m)(x-m-1)\cdots 1}$$

$$= N^m \sum_{y=0}^{\infty} \frac{e^{-N} N^y}{y!} \quad (y = x-m)$$

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5.1 c)

$$N = \langle x \rangle$$

$$N^2 = \langle x^2 - x \rangle = \langle x^2 \rangle - \langle x \rangle$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= N^2 + N - N^2$$

$$= N$$

$\boxed{5.2}$ In photon counting experiments, the relative uncertainity follows the relationship derived above.

$$\frac{\sigma}{\langle x \rangle} = \frac{1}{\sqrt{N}}$$

If we require a relative uncertainity of 1% (0.01), we can determine the minimum ~~ad~~ number of photons needed.

$$\frac{1}{\sqrt{N}} = 0.01$$

Solving for N - $\sqrt{N} = \frac{1}{0.01} = 100$

$$N = 10{,}000$$

For a relative uncertainity of $10^{-6}$ (one part per million), I apply the same relationship.

$$\frac{1}{\sqrt{N}} = 10^{-6}$$

Solving for N: $\sqrt{N} = 10^6$

$$N = 10^{12}$$

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To calculate the energy required for a photon counting measurement, I'll use the relationship between photon energy and wavelength.

$$E_{photon} = \frac{hc}{\lambda}$$

where $h = 6.626 \times 10^{-34}$ J.s (Planck's constant) – $c = 3 \times 10^{8}$ m/s (speed of light) – $\lambda = 600$ nm (wavelength for visible light)

For a single photon:

$$E_{photon} = \frac{(6.626 \times 10^{-34})(3 \times 10^{8})}{600 \times 10^{-9}} = 3.313 \times 10^{-19} \text{ J}$$

For the two cases calculated above.

1- For $N = 10^4$ photons (1% precision).

$$E_{total} = 10^4 \times 3.313 \times 10^{-19} = 3.313 \times 10^{-15} \text{ W.s}$$

2- For $N = 10^{12}$ photons (one part per million precision):

$$E_{total} = 10^{12} \times 3.313 \times 10^{-19} = 3.313 \times 10^{-7} \text{ W.s}$$

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5.3 a) Johnson-Nyquist Noise represents the thermal fluctuations in electronic components. The mean square voltage fluctuation across a resistor is given by

$$\langle V_{noise}^2 \rangle = 4kTR\,\Delta f$$

Where $k = 1.38 \times 10^{-23}$ J/K (Boltzmann constant). $T$ is temperature in Kelvin. $R$ is resistance in ohms. $\Delta f$ is bandwidth in Hz.

For a 10 kΩ resistor at room Temperature (300K) with a bandwidth of 20 kHz:

$$\langle V_{noise}^2 \rangle = 4 \times (1.38 \times 10^{-23}) \times 300 \times (10 \times 10^{3}) \times (20 \times 10^{3})$$

Breaking this down step by step:

$$4 \times 1.38 \times 10^{-23} = 5.52 \times 10^{-23}$$

$$5.52 \times 10^{-23} \times 300 = 1.656 \times 10^{-20}$$

$$1.656 \times 10^{-20} \times 10 \times 10^{3} = 1.656 \times 10^{-16}$$

$$1.656 \times 10^{-16} \times 20 \times 10^{3} = 3.312 \times 10^{-12}\ V^2$$

Taking the square root to find the RMS noise voltage:

$$V_{noise,rms} = \sqrt{3.312 \times 10^{-12}} = 1.82 \times 10^{-6}\ V = 1.82\ \mu V$$

The signal-to-noise ratio (SNR) is defined as:

$$SNR = 10 \log_{10} \left( \frac{\langle V_{signal}^2 \rangle}{\langle V_{noise}^2 \rangle} \right) dB$$

For a desired SNR of dB with the noise calculated above:

$$20 = 10 \log_{10} \left( \frac{\langle V_{signal}^2 \rangle}{3.312 \times 10^{-12}} \right)$$

Solving for $\langle V_{signal}^2 \rangle$:

$$\frac{20}{10} = \log_{10} \left( \frac{\langle V_{signal}^2 \rangle}{3.312 \times 10^{-12}} \right)$$

$$2 = \log_{10}\left(\frac{\langle V^2_{signal}\rangle}{3.312 \times 10^{-12}}\right)$$

$$10^2 = \frac{\langle V^2_{signal}\rangle}{3.312 \times 10^{-12}}$$

$$\langle V^2_{signal}\rangle = 3.312 \times 10^{-12} \times 100 = 3.312 \times 10^{-10} V^2$$

Taking the square root to find the RMS signal voltage:

$$V_{signal,rms} = \sqrt{3.312 \times 10^{-10}} = 1.82 \times 10^{-5} V = 18.2 \mu V$$

This calculation demonstrates the minimum signal level required to achieve reliable detection above the thermal noise floor, a critical parameter in designing sensitive electronic instruments.

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5.3 b) The thermal energy ~~considered~~ associated with a capacitor is related to its capacitance through the equipartition theorem:

$$\frac{1}{2} C \langle V^2 \rangle = \frac{1}{2} kT$$

Where - $C$ is capacitance - $\langle V^2 \rangle$ is the mean square voltage fluctuation - $k$ is Boltzmann's constant - $T$ is temperature in kelvin.

To find a capacitor with voltage fluctuations matching the Johnson noise calculated in part (a), I set:

$$\langle V^2 \rangle = 3.312 \times 10^{-12}\ V^2$$

Rearranging the equipartition equation:

$$C = \frac{kT}{\langle V^2 \rangle}$$

Substituting the values:

$$C = \frac{(1.38 \times 10^{-23})(300)}{3.312 \times 10^{-12}}$$

Breaking this down step by step:

$$1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21}$$

$$C = \frac{4.14 \times 10^{-21}}{3.312 \times 10^{-12}} = 1.25 \times 10^{-9}\ F = 1.25\ nF$$

5.3c) Shot noise arises from the discrete nature of electric charge and is described by:

$$\langle I^2_{noise} \rangle = 2qI\Delta f$$

Where $q = 1.602 \times 10^{-19}$ C (elemantary charge) - $I$ is the average current - $\Delta f$ is the band with.

For the RMS shot noise to be equal to 1% of the current:

$$\frac{I_{noise, rms}}{I} = 0.01$$

Squaring both sides

$$\frac{\langle I^2_{noise} \rangle}{I^2} = 0.0001$$

Substituting the shot noise formula:

$$\frac{2qI\Delta f}{I^2} = 0.0001$$

Solving for $I$:

$$I = \frac{2q\Delta f}{0.0001}$$

Substituting the values:

$$I = \frac{2 \times (1.602 \times 10^{-19}) \times (20 \times 10^{3})}{0.0001}$$

Breaking this down step by step:

$$2 \times 1.602 \times 10^{-19} = 3.204 \times 10^{-19}$$

$$3.204 \times 10^{-19} \times 20 \times 10^{3} = 6.408 \times 10^{-15}$$

$$I = \frac{6.408 \times 10^{-15}}{0.0001} = 6.408 \times 10^{-11} \text{ A} = 64.08 \text{ pA}$$

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