Optics Problem Set

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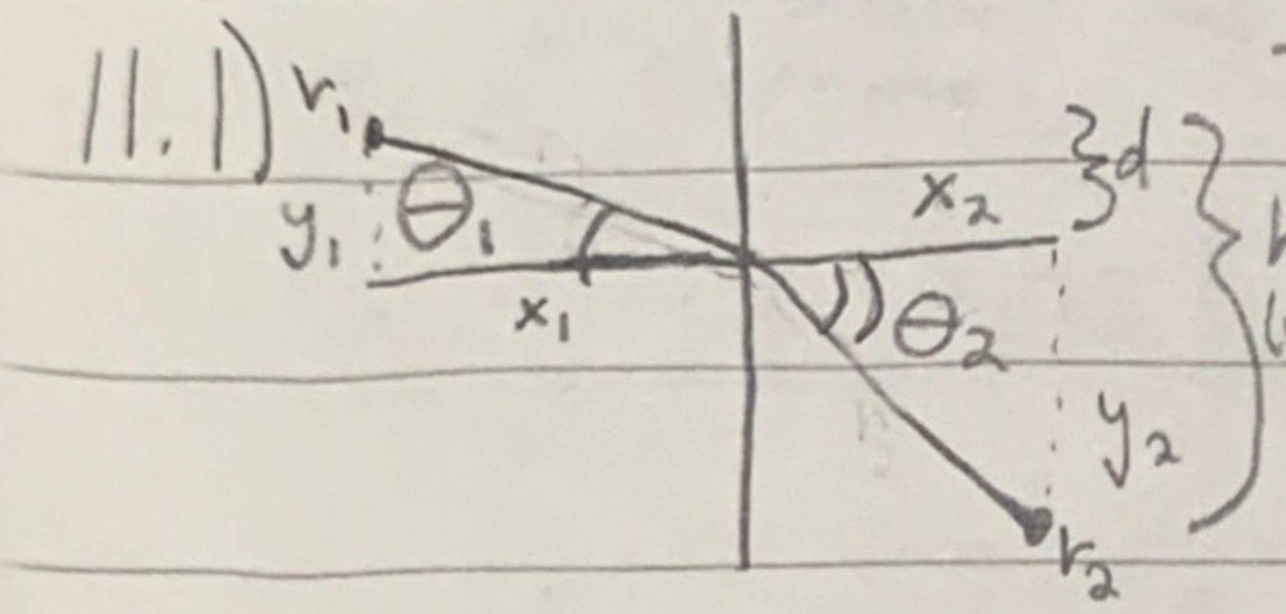
11.1)



The total time needed to travel from r_1 to r_2:

$$t = \frac{\sqrt{x_1^2+y_1^2}}{v_1} + \frac{\sqrt{x_2^2+y_2^2}}{v_2}$$

where v_1 & v_2 are the speeds of light in the materials.

Redefining t in terms of d since t is minimized by Fermat's principle:

$$t = \frac{\sqrt{x_1^2+d^2}}{v_1} + \frac{\sqrt{x_2^2+(h-d)^2}}{v_2}$$

Solving for the minimum travel time (where $\frac{dt}{dx}=0$):

$$0 = \frac{dt}{dx} = \frac{1}{2}\frac{(x_1^2+d^2)^{-1/2}}{v_1}\cdot 2d + \frac{1}{2}\frac{(x^2+(h-d)^2)^{-1/2}}{v_2}\cdot 2(h-d)\cdot(-1)$$

$$0 = \frac{d}{v_1\sqrt{x_1^2+d^2}} - \frac{h-d}{v_2\sqrt{x_2^2+(h-d)^2}}$$

$$0 = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \longrightarrow \frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

Plug in $v_1 = \frac{c}{n_1}$ & $v_2 = \frac{c}{n_2}$ (speed in terms of refractive index):

$$\frac{c}{n_1}\cdot\frac{n_2}{c} = \frac{\sin\theta_1}{\sin\theta_2} \longrightarrow n_1\sin\theta_1 = n_2\sin\theta_2$$

Snell's Law is confirmed to be $n_1\sin\theta_1 = n_2\sin\theta_2$

11.2a) Fresnel's Equations (10.20 & 10.24):

s-polarized & transmitted $E_\perp = \frac{2\sin\theta_2\cos\theta_0}{\sin(\theta_2+\theta_0)}E_0$, & $E_\parallel = \frac{2\cos\theta_0\sin\theta_2}{\sin(\theta_0+\theta_2)\cos(\theta_0-\theta_2)}E_0$, p-polarized & transmitted

s-polarized & reflected $E_\perp = \frac{\sin(\theta_2-\theta_0)}{\sin(\theta_2+\theta_0)}E_0$

(no reflection if materials are the same or when TX & RX are $\perp \rightarrow \theta_0=\theta_2$ or $\theta_0+\theta_2=\frac{\pi}{2}$)

$E_\parallel = \frac{\tan(\theta_0-\theta_2)}{\tan(\theta_0+\theta_2)}E_0$ p-polarized & reflected needs a k too. (but cancels out anyway)

Poynting vector: $S = \vec{H} \times \vec{E}$

TEM wave $\therefore$ $S = |\vec{H}||\vec{E}|\hat{k}$ & $|\vec{H}| = \frac{1}{\omega\mu_0}\hat{k} \times \vec{E} = \frac{1}{\mu_m} E$

$\therefore$ $S = \frac{1}{\omega\mu_0} \cdot E^2 \hat{k}$

$$\langle S \rangle_{in} = \frac{1}{\omega\mu_0} \langle E^2 e^{-2i(\vec{k}\cdot\vec{r}+\omega t)} \rangle = \frac{1}{2\omega\mu_0} E_0^2$$

$$\langle S \rangle_{reflected} = \frac{1}{\omega\mu_0} \left(E_\perp^{R2} + E_\parallel^{R2} \right) \text{ (avg of two polarizations)}$$

$$= \frac{1}{\omega\mu_0} E_0^2 \left(\frac{\sin^2(\theta_2-\theta_0)}{\sin^2(\theta_2+\theta_0)} + \frac{\tan^2(\theta_0-\theta_2)}{\tan^2(\theta_0+\theta_2)} \right)$$

$$\langle S \rangle_{transmitted} = \frac{1}{\omega\mu_0} \left(E_\perp^{T2} + E_\parallel^{T2} \right)$$

$$= \frac{1}{\omega\mu_0} E_0^2 \left(\frac{4\sin^2\theta_2 \cos^2\theta_0}{\sin^2(\theta_2+\theta_0)} + \frac{4\sin^2\theta_2 \cos^2\theta_0}{\sin^2(\theta_2+\theta_0)\cos^2(\theta_2-\theta_0)} \right)$$

$$\therefore R = \left(\frac{\sin(\theta_2-\theta_0)}{\sin(\theta_2+\theta_0)} \right)^2 + \left(\frac{\tan(\theta_2-\theta_0)}{\tan(\theta_2+\theta_0)} \right)^2$$

(s-polarized) (p-polarized)

$$T = \left(\frac{2\sin\theta_2 \cos\theta_0}{\sin(\theta_2+\theta_0)} \right)^2 + \left(\frac{2\sin\theta_2 \cos\theta_0}{\sin(\theta_2+\theta_0)\cos(\theta_2-\theta_0)} \right)^2$$

(s-polarized) (p-polarized)

b) At normal incidence: $R = \left(\frac{n_1-n_2}{n_1+n_2} \right)^2 = \left(\frac{1.5-1}{1.5+1} \right)^2 = 0.04$

c) $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1} \right)$

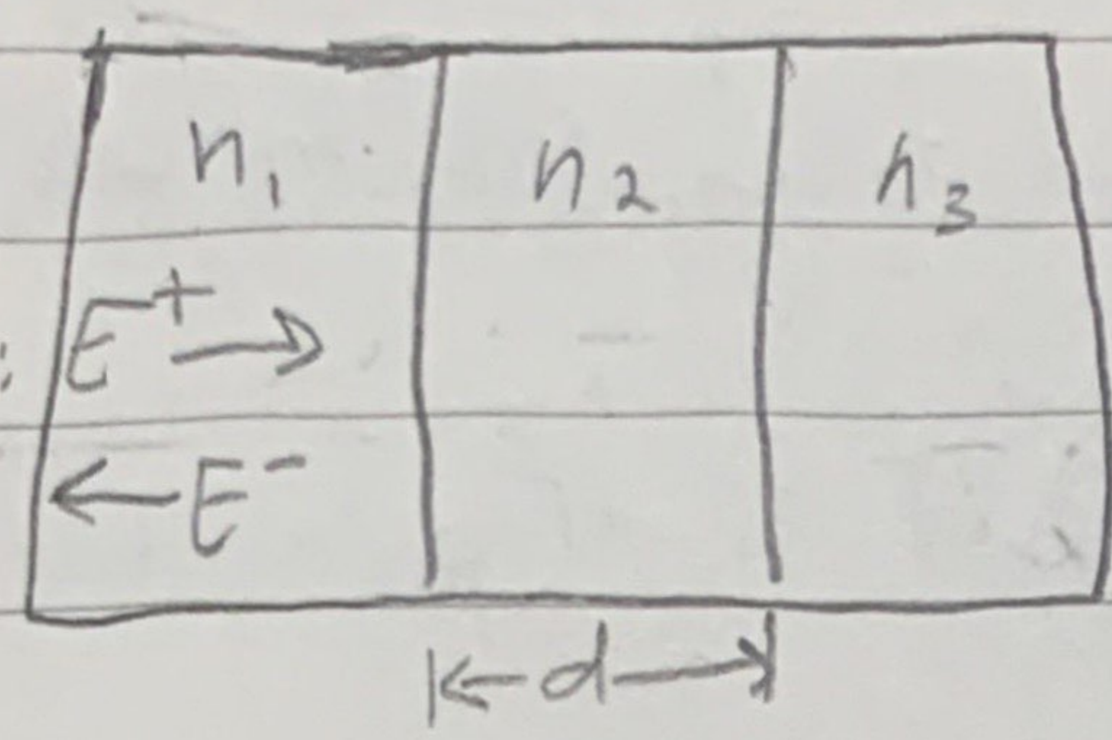
For glass-air interface, $n_1 = 1$ & $n_2 = 1.5$ $\therefore$ $\theta_B = 56.3°$

For air-glass interface, $n_1 = 1.5$ & $n_2 = 1$ $\therefore$ $\theta_B = 33.7°$

d) critical angle $\theta_c = \sin^{-1}\left(\frac{n_1}{n_2} \right) \rightarrow$ For glass-air interface, $\theta_c = 41.8°$

11.3a)

Figure 10.6:



$$R_{ij} = \frac{n_i - n_j}{n_i + n_j} \quad \text{(reflection)}$$

$$T_{ij} = \frac{2n_i}{n_i + n_j} \quad \text{(transmission)}$$

$$\varphi = e^{ik_j d} \quad \text{(phase change)}$$

Total reflection intensity:

$$E_R = E_0\left(R_{12} + T_{12} R_{23} e^{ik_2 2d} T_{21} + T_{12} R_{23} e^{ik_2 2d} R_{21} R_{23} e^{ik_2 2d} T_{21} + T_{12} R_{23} e^{ik_2 2d} R_{21} R_{23} e^{ik_2 2d} R_{21} R_{23} e^{ik_2 2d} T_{21} + \ldots\right)$$

equivalently

$$E_R = E_0\left(R_{12} + T_{12} R_{23} e^{ik_2 2d} T_{21} \sum_{n=0}^{\infty} \left(R_{21} R_{23} e^{ik_2 2d}\right)^n\right)$$

Simplify using $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$:

$$E_R = E_0\left(R_{12} + \frac{T_{12} R_{23} e^{ik_2 2d} T_{21}}{1 - R_{21} R_{23} e^{ik_2 2d}}\right) \quad \text{where } R_{12} = -R_{21}$$

$$E_R = E_0\left(\frac{R_{12} + R_{12}^2 R_{23} e^{ik_2 2d} + T_{12} R_{23} e^{ik_2 2d} T_{21}}{1 + R_{12} R_{23} e^{ik_2 2d}}\right)$$

$$= E_0\left(\frac{R_{12} + R_{23} e^{ik_2 2d}\left(R_{12}^2 + T_{12} T_{21}\right)}{1 + R_{12} R_{23} e^{ik_2 2d}}\right)$$

Simplify in parts: $R_{12}^2 + T_{12} T_{21} = \frac{(n_1 - n_2)^2}{n_1 + n_2^2} + \frac{4 n_1 n_2}{(n_1 + n_2)^2}$

$$\hookrightarrow = \frac{n_1^2 - 2n_1 n_2 + n_2^2 + 4 n_1 n_2}{n_1^2 + 2 n_1 n_2 + n_2^2} = 1$$

$$E_R = E_0\left(\frac{R_{12} + R_{23} e^{ik_2 2d}}{1 + R_{12} R_{23} e^{ik_2 2d}}\right)$$

Plug in: $R = \frac{|E_R|^2}{|E_0|^2} = \left(\frac{R_{12} + R_{23} e^{ik_2 2d}}{1 + R_{12} R_{23} e^{ik_2 2d}}\right)^2$

11.3b) The reflection disappears during destructive interference:

$$\phi = \frac{4k n_2 d}{\lambda} = (2m+1)\pi \quad \text{(phase shift)}$$

$$\therefore d = \frac{(2m+1)\lambda}{4n_2}$$

For maximum thickness condition, $m=0$:

$$\therefore \quad d = \frac{\lambda}{4n_2}$$

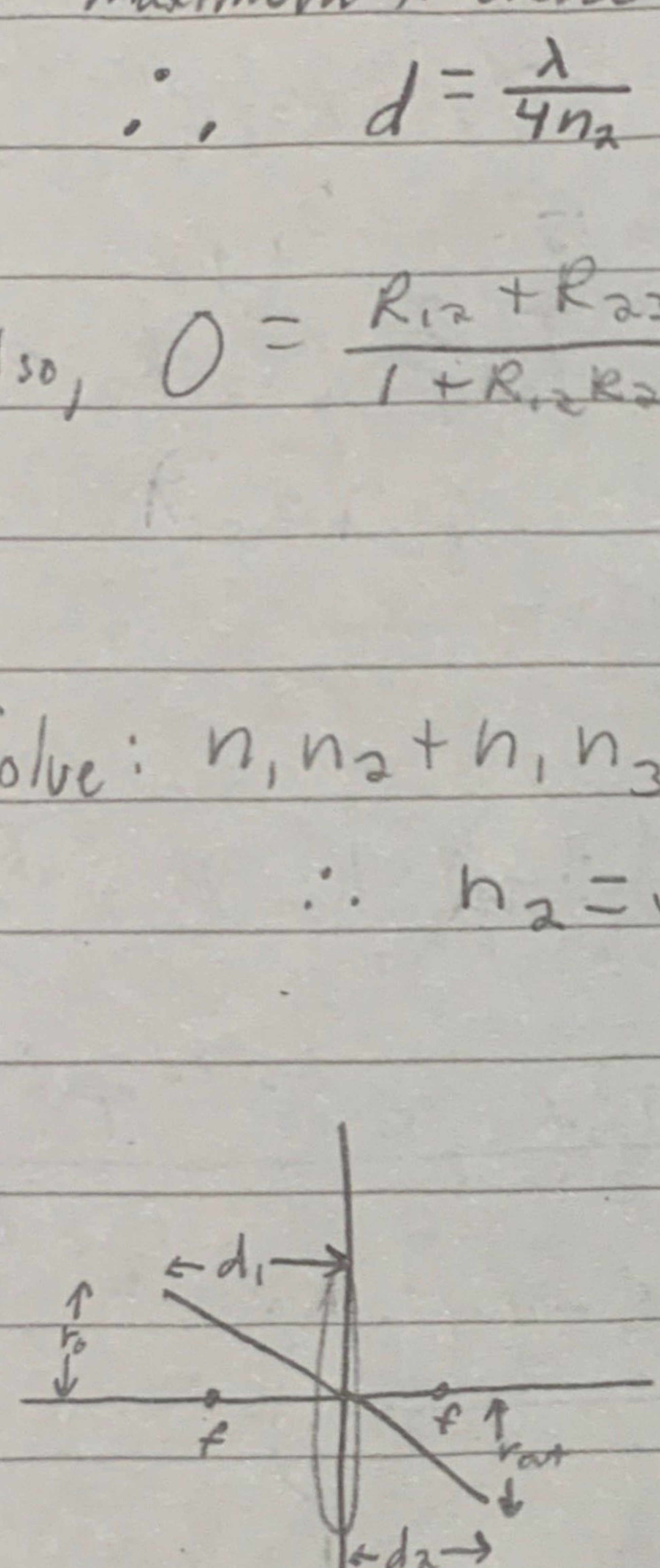
Also, $0 = \frac{R_{12} + R_{23}}{1 + R_{12}R_{23}} \rightarrow R_{12} = R_{23}$

$$\frac{n_1 - n_2}{n_1 + n_2} = \frac{n_2 - n_3}{n_2 + n_3}$$

Solve: $n_1 n_2 + n_1 n_3 - n_2^2 - n_2 n_3 = n_1 n_2 - n_1 n_3 + n_2^2 - n_2 n_3$

$$\therefore \quad n_2 = \sqrt{n_1 n_3}$$

11.4)



(using ray matrices)

$$M_{tot} = M_{prop2} \cdot M_{lens} \cdot M_{prop}$$

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r'_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_1 \\ -1/f & 1 - d_1/f \end{bmatrix}$$

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} 1 - d_2/f & d_1 + d_2(1 - d_1/f) \\ -1/f & -d_1/f \end{bmatrix} \begin{bmatrix} r_0 \\ r'_0 \end{bmatrix}$$

For all rays to converge on the image plane; r_0' must $= 0$ (This makes it independent of angle)

$$\therefore d_1 + d_2(1 - d_1/f) = 0$$

Solve for d_2: $d_2 = \frac{d_1 f}{d_1 - f} \xrightarrow{\text{thin lens eqn}} \frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$ (Eqn. 10.35)

$\therefore$ All rays will meet at the image plane at a distance d_2 from the thin lens.

Looking at magnification: $M = \frac{r_0'}{r_0} = A = 1 - d_0/f$ (magnification)

11.5a) divergence angle $\Theta = \frac{\lambda}{\pi n w}$ where $\lambda = 790\,nm$

$$\Theta = \frac{790 \cdot 10^{-9}}{\pi \cdot 1 \cdot 10^{-6}}$$

Beam divergence angle is $\Theta = 0.251$ rad $\rightarrow$ $\Theta = 14.4°$

11.5b) $\lambda_{new} = \lambda \cdot \frac{D_{new}}{D}$ (also $\lambda = \Theta \cdot \pi n w = 0.251 \cdot \pi \cdot 1 \cdot 10^{-6}$)

$$\lambda_{new} = 790 \cdot \frac{1 \times 10^{-7}}{1 \cdot 10^{-6}} \rightarrow \lambda_{new} = 79\,nm$$

11.5c) $\Theta = \frac{600\,nm}{\pi \cdot 1 \cdot 10\,cm} = 1.9 \cdot 10^{-5}$ rad where 10cm = license plate size (minimum resolvable)

Knowns: $\lambda = 600\,nm$, $h = 0.1\,m$, $L = 200\,km = 2 \cdot 10^{5}\,m$

$$r = h \tan\Theta \approx h\Theta$$

$$\frac{h}{L} > \frac{1.22\lambda}{D}$$

$$r = (200\,km)(1.9 \cdot 10^{-5}) = 3.8\,m$$

SHOULD BE 3.8m

Plug in and solve for D: $D > 1.46\,m$

$= 3.8\,m$

11.6) Periodically forced Lorentz model to find dielectric constant as a function of f

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = -eE_0 e^{-i\omega t}$$ where $E_0 e^{-i\omega t} = E(t)$

Assuming a steady state response as in $x = x_0 e^{-i\omega t}$, solve for x_0:

$$x_0 = \frac{-eE_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

Solve for induced polarization: $P = Np$ (dipole moment / unit volume)

Since $p = -ex$ $\therefore$ $P = -Nex$

Substitute x_0 into P:

$$P = -Nex = Ne^2\left(\frac{E_0}{m(\omega_0^2-\omega^2-i\gamma\omega)}\right)$$ where N is #oscillators/unit volume

Use the expression for the dielectric constant:

$$D = \varepsilon_0 E + P$$

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{Ne^2}{\varepsilon_0 m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}\right)$$

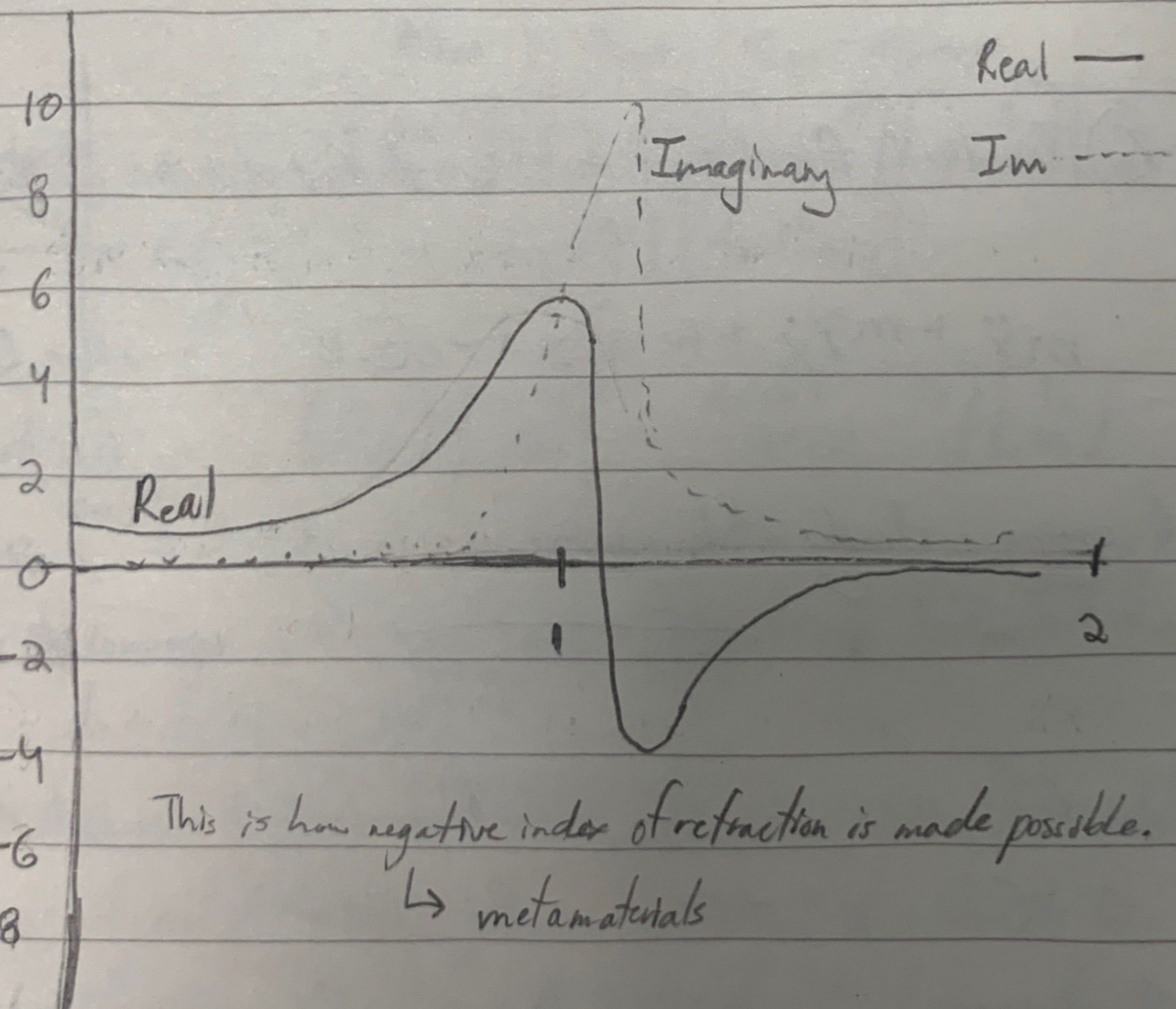
plasma frequency: $\omega_p^2 = \frac{Ne^2}{\varepsilon_0 m}$

$$\varepsilon(\omega) = \varepsilon_0\left(1 + \frac{\omega_p^2}{\omega_0^2-\omega^2-i\gamma\omega}\right)$$

wherein $Re(\omega)$ is dispersion & $Im(\omega)$ is absorption loss

This is plotted as Figure 10.5 Anomalous dispersion.

Roughly:



This is how negative index of refraction is made possible.
↳ metamaterials