PS2

Wednesday, February 19, 2025 9:30 PM

4.1 **Q1)** $H(p) = -\sum_{i=1}^{X} p_i \log p_i$

log $\to$ continuous

$H(p)$ is a linear combination of continuous functions.

Hence continues

$p_i \geq 0$, $p_i \log p_i \leq 0$
< 1

$$H(p) = -\sum_i p_i \log p_i \geq 0$$

if any $P = 1$ then $H(p) = 0$

$$H(p) \geq 0$$

Entropy is max when $p_i = 1/X$ for

thus, $H(p) \leq \log(x)$ Bounded

for x & y independent variables:

$$H(x,y) = H(x) + H(y)$$

$p(x,y) = p(x)\,p(y)$

$\hookrightarrow H(x,y) = -\sum_{x,y} p(x)p(y) \log(p(x)p(y))$

$= -\sum p(x)p(y) \log p(x) - \sum p(x)\,p(y) \log p(y)$

$= \sum \underbrace{p(y)}_{1} \times H(x) + \sum \underbrace{p(y)}_{1} \times H(y)$

$= \underline{H(x) + H(y)} \to$ Independence.

Q2) $I(x,y) = H(x) + H(y) - H(x,y)$

$= H(y) - H(y/x) = H(x) - H(x/y)$

$$= H(y) - H(y/x) = H(x) - H(x/y)$$

$$I(x,y) = \sum p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

$$\underline{H(x,y) = H(x) + H(y/x) = H(y) + H(x/y)}$$

Mutual info is (info separately) − (info together)

$$= H(x) + H(y) - H(x,y)$$

$$\cancel{H(x)} + H(y) - \cancel{H(x)} - H(y/x)$$
$$= \underline{H(y) - H(y/x)}$$

$$H(x) + \cancel{H(y)} - \cancel{H(y)} - H(x/y)$$
$$= \underline{H(x) - H(x/y)}.$$

Q3) Majority voting – 3 bits

P(correct) = P(2 correct, 1 wrong) + P(3 correct)
exclusively

(a) $= 3\epsilon(1-\epsilon)^2 + (1-\epsilon)^3$ 3 bits

$\hookrightarrow \epsilon'$

successive maj. voting

$= 3\epsilon'(1-\epsilon')^2 + (1-\epsilon')^3$ 6 bits

$\hookrightarrow \epsilon''$

↓

keeps recursively progressing.

3 times $= \underline{3\epsilon''(1-\epsilon'')^2 + (1-\epsilon'')^3}$ 9 bits

⋮

$3\epsilon^{n}(1-\epsilon^{n'})^2 + (1-\epsilon^{n'})^3$ 3^n bits

Q4) gaussian distribution $p(x) = \frac{1}{\;} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\frac{(x-\mu)^2}{}$

Q4) gaussian distribution $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$H(x) = -\int p(x) \log p(x)\, dx$$

$$H(x) = -\int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \times \left[\frac{-1}{2}\log 2\pi\sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$\log p(x) = \frac{-1}{2}\log 2\pi\sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}$$

$$H(x) = \frac{\log 2\pi\sigma^2}{2} \cancel{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}} + \int \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}} \times \frac{(x-\mu)^2}{\sigma^2} dx$$

1 for gaussian

$$\frac{1}{2\sigma^2} \int \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}}_{p(x)} (x-\mu)^2 dx$$

$$\Rightarrow H(x) = \frac{\log 2\pi\sigma^2}{2} + \frac{1}{2\sigma^2} \int p(x)(x-\mu)^2 dx$$

definition of σ^2

$$H(x) = \frac{1}{2}\left(1 + \log(2\pi\sigma^2)\right)$$

Q5) $SNR = 20\,dB$ since SNR is for power.

$$= 10^{20/10} = \underline{100}$$

$$C = \Delta f \log_2\left(1 + \frac{S}{N}\right) = 3300 \log_2(101)$$

$$\approx \underline{22k \text{ bits/s}}.$$

for 1Gbps

$$10^9 = 3300 \log_2(1 + SNR)$$

$$1 + SNR \approx 2^{3 \times 10^5}$$

$$SNR \approx 2^{3 \times 10^5}$$ too high

Q6) $Var(f(x)) = Var\left(\frac{1}{n}\sum x_i\right)$

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$$= \frac{1}{n^2} Var\left(\sum x_i\right) = \frac{1}{n^2}\left(\sum \sigma^2\right) = \frac{n\sigma^2}{n^2}$$

$$\underline{\underline{Var = \frac{\sigma^2}{n}}}$$