4.2

break this apart

$$I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)\,p(y)}$$

$$= \sum_x \sum_y p(x,y) \cdot \log(p(x,y)) - \sum_x p(x,y) \cdot \log p(x)$$

$$- \sum_y p(x,y) \cdot \log p(y)$$

drop the summations with no terms present

4.4

$H(x,y) =$

$-\sum_x \sum_y p(x,y) \cdot \log p(x,y)$

$$= H(x,y) - H(x) - H(y)$$

so signs flip

by the def. of conditional entropy

$$H(y|x) = H(x,y) - H(x) \Rightarrow H(x,y) = H(y|x) + H(x)$$

$$H(x|y) = H(x,y) - H(y) \Rightarrow H(x,y) = H(x|y) + H(y)$$

can be plugged into above

4.3 a. Using majority voting:

ϵ = prob. of single channel error

P_{error} = P(at least 2 wrong bits)

$= 3 \cdot P(2 \text{ wrong}) + P(3 \text{ wrong})$

Note that $P(3 \text{ wrong}) = \epsilon^3$

Binomial dist

3 is from choose 2

$= 3(1-\epsilon)\epsilon^2 + \epsilon^3$

$(1-\epsilon)$: prob of getting 1 right

$= 3\epsilon^2 - 3\epsilon^3 + \epsilon^3$

$= 3\epsilon^2 - 2\epsilon^3$

$\epsilon^2 = P(2 \text{ wrong})$

b. $P_{error} = 3\epsilon_1^2 - 2\epsilon_1^3$

Nest the probs. for error

where $\epsilon_1 = 3\epsilon^2 - 2\epsilon^3$

$= 3(3\epsilon^2 - 2\epsilon)^2 - 2(3\epsilon^2 - 2\epsilon)^3$

c. each round of voting triples # sent

so n rounds requires 3^n bits

so we have

$P(1 \text{ right}) \cdot P(2 \text{ wrong})$

$\cdot 3$

can happen 3 ways

Not sure how to write this recursion mathematically but when plotted

4.4 Diff. Entropy $= \int_{-\infty}^{\infty} p(x) \log p(x)\, dx$

where for a gaussian process $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

where μ = mean

σ^2 = variance

substitute

$x = x - \mu^2$

diff entropy

$$= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \cdot \log\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right] dx$$

the $\frac{1}{\text{sqrt}}$ is pulled out as a $-1/2$

choose nat. log to make the math nicer.

$$= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x)^2}{2\sigma^2}} \cdot \left[-\frac{1}{2}\log(2\pi\sigma^2) - \frac{x^2}{2\sigma^2}\right] dx$$

THIS is p(x)! integrated from $-\infty$ to ∞ is $=1$

$$= \frac{1}{2}\log(2\pi\sigma^2) \cdot \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\left(e^{\frac{-(x)^2}{2\sigma^2}} (dx)\right) + \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\left(e^{\frac{-(x)^2}{2\sigma^2}}\right)\left(\frac{x^2}{2\sigma^2}\right) dx$$

pull out

$$= -\frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{x^2}{\sigma^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\left(e^{\frac{-x^2}{2\sigma^2}}\right) dx$$

can't do the same trick

4.5 a. Telephone line $\Delta f = 3300$ Hz

$SNR = 20$ dB

what is capacity (in bits / second)

Shannon-Hartley theorem

$$= \Delta f \log_2 \left(1 + \frac{S}{N}\right)$$

$$= 3300 \cdot \log_2 (1 + 10^2)$$

$$= 3300 \log_2 (101)$$

$$= 3300 \cdot 6.6 = 21945 \text{ bits/s}$$

b. $1 \cdot 10^9$ bits/s $= 3300 \cdot \log_2 (1 + S/N)$

$$\log_2 \left(1 + \frac{S}{N}\right) = 3.03 \cdot 10^5$$

$$\log_{10} (1 + S/N) = 3.03 \cdot 10^5 \cdot \log_{10} 2$$

$$= 91212$$

$$\approx 10^5$$

$$10 \log (S/N) = 10^6 \text{ dB}$$

$SNR = 10^6$ db

4.6

$$f(x_1, x_2 \dots x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

mean

$$\langle f(x_1, x_2 \dots x_n)\rangle = \langle \frac{1}{n}\sum_{i=1}^{n} x_i \rangle = \frac{1}{n}\sum_{i=1}^{n} \langle x_i \rangle$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_0$$

$$\sum_{i=1}^{n} x_0 = \underbrace{x_0 + x_0 + x_0 \dots}_{n \text{ times}}$$

$$= n \cdot x_0$$

$$= \frac{1}{n} \cdot n \cdot x_0 = x_0$$

Cramér-Rao Bound states that the variance of the estimator must be $\geq$ the inverse of the fisher information

4.1 continuity