6.1 prove $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

BAC-CAB or Lagrange's Formula or

Triple product expansion

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

etc...

FIRST WE TAKE $\vec{b} \times \vec{c}$ (inside the parenthesis)

alternating signs

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \hat{i}(b_y c_z - b_z c_y)$$

$$- \hat{j}(b_x c_z - b_z c_x)$$

$$+ k(b_x c_y - b_x c_y)$$

det

THEN WE CROSS THAT WITH $\vec{A}$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & b_z c_x - b_x c_z & b_x c_y - b_y c_x \end{vmatrix}$$

$$= \hat{i}(a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z)$$

rather than do it all out...

$$\hat{\imath}\Big(a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z + a_x b_x c_x$$
$$- a_x b_x c_x\Big)$$

add + subtract the same term then factor out by

$$= b_x\underbrace{(a_x c_x + a_y c_y + a_z b_z)} - c_x\underbrace{(a_y b_y + a_z b_z + a_x b_x)}$$

$$\hat{\imath}\left[\ (\vec{a}\cdot\vec{c})b_x \quad - \quad (\vec{a}\cdot\vec{b})c_x\ \right]$$

makes sense as they're 3x1 vectors so...

$$\hat{\imath}(\vec{a}\cdot\vec{c})b_x - \hat{\imath}(\vec{a}\cdot\vec{b})c_x$$

$$+\hat{\jmath}(\vec{a}\cdot\vec{c})b_y - \hat{\jmath}(\vec{a}\cdot\vec{b})c_y$$

we can conclude for $\hat{\jmath}$ + $\hat{k}$

$$+\hat{k}(\vec{a}\cdot\vec{c})b_z - \hat{k}(\vec{a}\cdot\vec{b})c_y$$

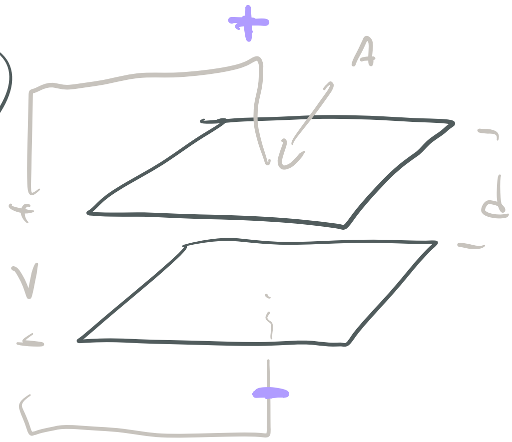
pull out the $(\vec{a}\cdot\vec{c})$

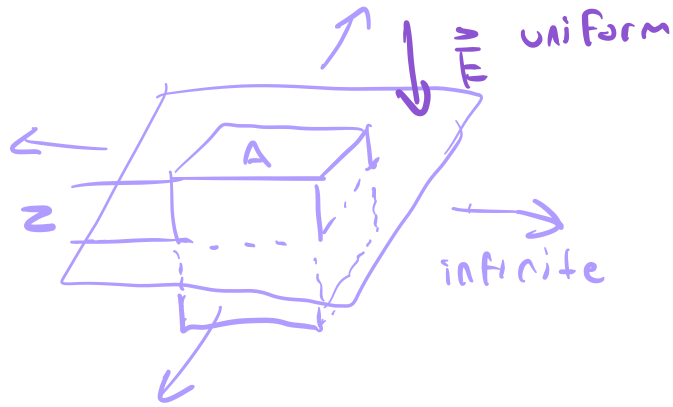
$$(\vec{a}\cdot\vec{c})\underbrace{(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k})}_{\vec{b}} = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$-(\vec{a}\cdot\vec{b})\underbrace{(c_x\hat{\imath} + c_y\hat{\jmath} + c_z\hat{k})}_{\vec{c}}$$

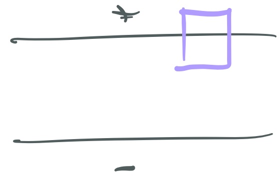
THEREFORE $\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \vec{E}\underbrace{(\nabla\cdot\nabla)}^{\nabla^2}$

6.2





6.2a



$C = \frac{Q}{V}$ charge / potential diff.

To solve for Q + V, we can model the cap first as a single infinite plate. To use Gauss' we'll consider a box bisecting the plate of area A + height 2z (equal above + below).

We know all the flux is traveling vertically through the box. So we can ignore flux through the sides, therefore the surface integral only needs to account for the top + bottom faces of area 2A.

$$\int_V \nabla \cdot \vec{E}\, dV = \int_S \vec{E}\, dA = \int_V \frac{\rho}{\varepsilon}\, dV$$

$$2A\vec{E} = \rho A / \varepsilon$$

mildly confused by jump from volume integral to area

solving for field strength: $E = \rho / 2\varepsilon$
which is uniform between the plates!

extrapolating to two plates with opposite charges we multiply by 2, so $E = \dfrac{\rho}{\varepsilon}$

we know from (6.30) that voltage $V =$

$$V = \int_0^d \vec{E} \cdot d\vec{l}$$ ← negative sign is relative?

$$V = \frac{\rho}{\varepsilon} \cdot d$$ where d is the distance between plates

charge = charge density · area = $\rho \cdot A$

so $C = \dfrac{\cancel{\rho} \cdot A \varepsilon}{\cancel{\rho} d} = \dfrac{A\varepsilon}{d}$

↑ purely geometric + material driven

6.2 b

displacement current is the time derivative of $\vec{D}$

$$\int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{A} = \varepsilon \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

displacement current is just the applied electric field $\vec{E} \cdot \varepsilon$

we know from a. that $E = \rho/\varepsilon$

$$\therefore \frac{\cancel{\varepsilon}}{\cancel{\varepsilon}} \int_S \frac{\partial \rho}{\partial t} d\vec{A}$$

charge density σ

$$\sigma = Q/A$$

$$\frac{1}{A} \int_S \frac{\partial Q}{\partial t} \partial \vec{A}$$

This is current!

change in charge over time

$\therefore$ displacement current $= \frac{\cancel{A}}{\cancel{A}} I = I$

6.2 c

potential energy

$$U = \frac{1}{2}\left(\underbrace{E \cdot D}_{} + \underbrace{B \cdot H}_{}\right)$$

$D = E\varepsilon$

no current flow (fixed potential) so magnetic field is zero.

where

$$U = \frac{\varepsilon}{2}\vec{E}\cdot\vec{E} \qquad \vec{E} = \frac{\vec{P}}{\varepsilon}$$

$$= \frac{\cancel{\varepsilon}}{2}\frac{\vec{P}}{\cancel{\varepsilon}}\cdot\frac{\vec{P}}{\varepsilon} = \frac{\vec{P}^2}{2\varepsilon}$$

now integrate over volume $A \cdot d$

$$= \frac{1}{2\varepsilon}\int_V P^2\,dV = \frac{P^2 A d}{2\varepsilon}$$

$P^2 = \vec{E}^2\varepsilon^2$

we know $C = \frac{A\varepsilon}{d}$

$$= \frac{C}{2}\frac{P^2 d^2}{\varepsilon^2} = \frac{C}{2}\vec{E}^2 d^2$$

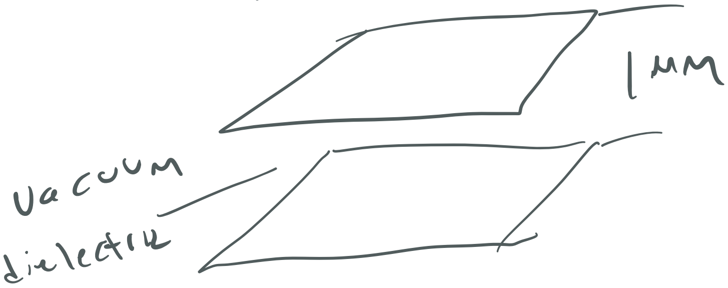
$\vec{E}^2 d^2 = V^2$

potential difference!

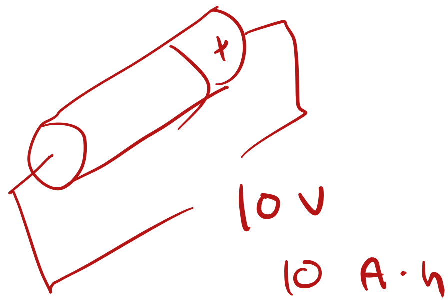
$$= \frac{CV^2}{2}$$

6.2 d   compare a battery & a capacitor

solve for A?



VS.



$C = \frac{A\varepsilon_0}{d}$

energy = $V \cdot I \cdot t$ (seconds)

$= 10 \cdot 10 \cdot 60 \cdot 60$

$U = 360{,}000\ J$

also from 6.2c $U = \frac{CV^2}{2}$

$U = \frac{A\varepsilon_0 V^2}{2d}$

very large cap!

$A = \frac{2dU}{V^2\varepsilon_0} = \frac{2(10^{-6}) \cdot 360 \cdot 10^3}{10^2 \cdot 8.85 \cdot 10^{-12}} = 8.07 \cdot 10^8\ m^2$

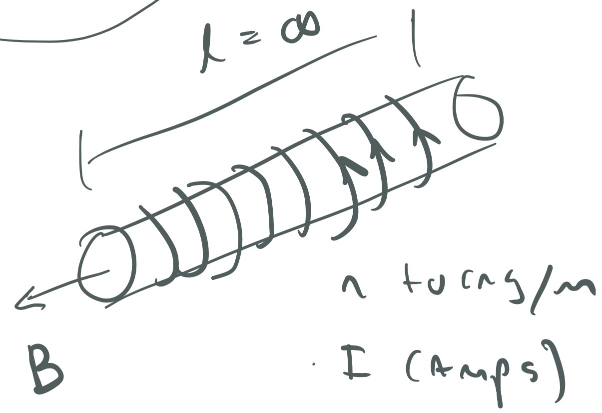
for 2nd part, assuming just the 1 μm dielectric thickness:

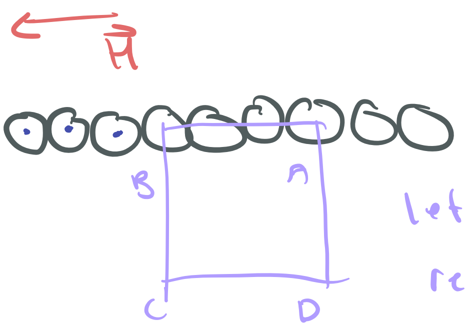
10 cm = .1 m · .1 m = $0.01\ m^2$ so...

$10^{-6}\left(\frac{8 \cdot 10^8}{0.01}\right) = 80{,}000\ m$

or 80 km

6.3 a





Stokes is perfect here because it tells us that the curl (mag field) $\int_S \nabla \times \vec{E}\, d\vec{A}$

is equal to the alignment of $\vec{E}$ with the bounding curve $\oint_{\ell} \vec{E}\, d\vec{\ell}$

let's make rectangle ABCD our bounding curve

$$\oint \vec{H} \cdot d\vec{\ell} = \int_A^B \vec{H}\, d\vec{\ell} + \int_B^C \overset{0}{\cancel{\vec{H}\, d\vec{\ell}}} + \int_C^D \overset{0}{\cancel{\vec{H}\, d\vec{\ell}}} + \int_D^A \overset{0}{\cancel{\vec{H}\, d\vec{\ell}}}$$

$$= H\ell$$

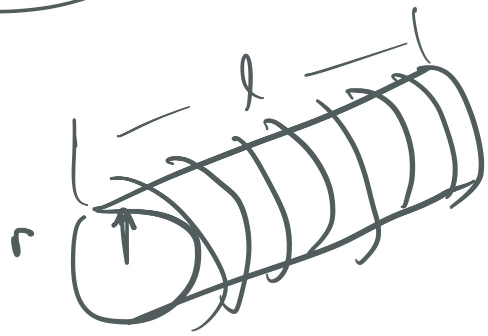
$$\int_S \nabla \times \vec{H} \cdot d\vec{A} = \int_S \vec{J} \cdot d\vec{A} = I \cdot \ell$$ over the full solenoid with n turns

curl of H field is current $\vec{J}$

$$= I \cdot \ell \cdot n$$

so: $H\cancel{\ell} = I \cdot \cancel{\ell} \cdot n$ $\quad H = In$

6.3 b



energy stored in the solenoid would be purely the magnetic field, so

$$U = \frac{1}{2}(\cancel{E \cdot D} + H \cdot B)$$

$$U = \frac{1}{2}\vec{H} \cdot \vec{B}$$

where $\vec{B} = \mu \vec{H}$

$$U = \frac{\mu}{2}\vec{H}^2$$

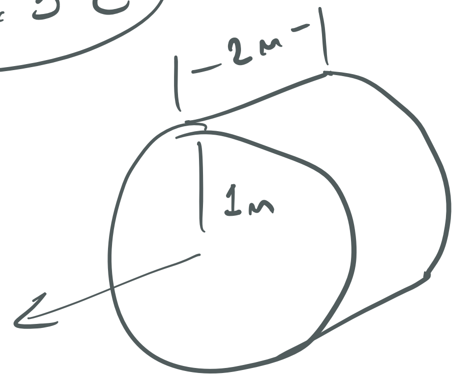
integrated over the volume of the solenoid

$$U = \frac{\pi r^2 l}{2}\vec{H}^2$$

plugging in $H = In$

$$U = \frac{\pi r^2 l I^2 n^2}{2}$$

6.3 c



$H = 10T$

$$U = \frac{1}{2} B \cdot H = \frac{1}{2} B \cdot B/\mu$$

$$= \frac{1}{2} B^2/\mu$$

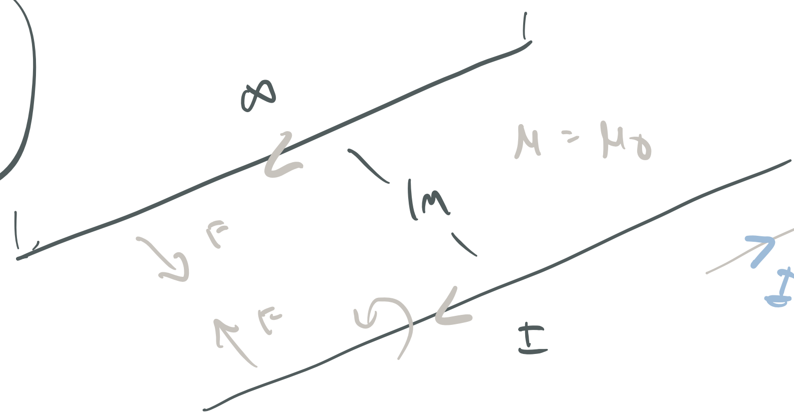
outward force $= \frac{\partial}{\partial r}\left(U(r)\right)$

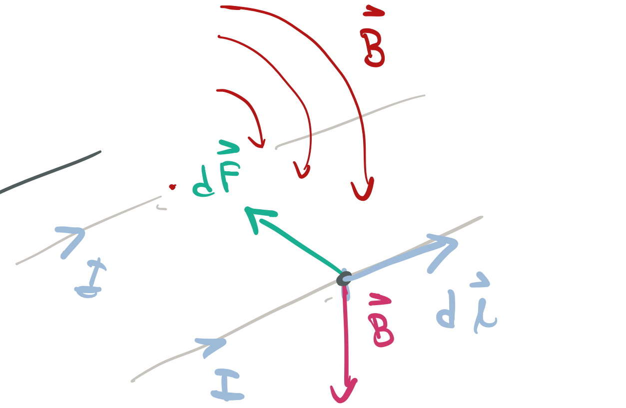
we have U_{total} so:

$$F = U_{total} \cdot \pi r^2 \cdot l = \frac{1}{2} \frac{B^2}{\mu} \pi r^2 l$$

$$= \frac{(10T)^2}{2 \cdot 1.257 \cdot 10^{-6}} \cdot \pi \cdot 1m^2 \cdot 2m = 2.5 \cdot 10^{8} \text{ N}$$

6.4





$d\vec{F} = I(d\vec{l} \times \vec{B})$

$$d\vec{F} = I\, d\vec{l} \times \vec{B}$$

$$= \mu_0 I\, d\vec{l} \times \vec{H}$$

sub in (6.87) $\vec{H} = \frac{I}{2\pi r}$

$$= \frac{\mu_0 I^2}{2\pi r}\, d\vec{l}$$

where for a 1m distance r

$$d\vec{F} = \frac{1.26e^{-6}(1A)^2}{2\pi(1m)^2}\, d\vec{l} = 2\cdot 10^{-7}\, N\, d\vec{l}$$

or $2\cdot 10^{-7}$ N / 1m length

this is the force exerted by the presence of wire X's flux $\vec{B}$ interacting with wire Y's current $I d\vec{l}$

so this is the total force not a 2x

(b) (impossible) very difficult to validate experimentally

6.6 a

1 kW/m^2 from sunlight

estimate associated $\vec{E}$

The Poynting vector represents the energy flux in electric + magnetic fields (their cross product)

$$\vec{P} = \vec{E} \times \vec{H}$$

for light in free space they're perpendicular, so

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin\theta \qquad \theta = 90^\circ$$

$$\|\vec{E} \times \vec{H}\| = \|\vec{E}\| \cdot \|\vec{H}\| \cdot (1)$$

using (6.105) $\dfrac{\|\vec{E}\|}{\|\vec{H}\|} = \left(\dfrac{\mu_0}{\varepsilon_0}\right)^{1/2} \Rightarrow \|\vec{H}\| = \|\vec{E}\| \left(\dfrac{\varepsilon_0}{\mu_0}\right)^{1/2}$

sub in $\|\vec{H}\|$

$$\|\vec{P}\| = \|\vec{E}\|^2 \left(\frac{\varepsilon_0}{\mu_0}\right)^{1/2}$$

we can relate $\vec{P}$ to $\dfrac{\text{power}}{m^2}$ using (6.119)

$$W = -\int_S \vec{P} \cdot d\vec{A}$$

$$P = -\frac{1000\,W}{m^2}$$

so: $\|\vec{E}\|^2 = 1000\,W/m^2 \left(\dfrac{8.85e^{-12}}{1.26e^{-6}}\right)^{-1/2}$

$$\|\vec{E}\| = 614.285 \ \frac{V}{m}$$

6.6 b

for 1 W laser focused to 1 mm^2

now we can plug + chug from part a:

$$\|\vec{E}\| = \sqrt{\frac{1\,W}{.001^2\,m}} = 19424\ V/m$$

much higher!

$1.9424 \cdot 10^4\ V/m$

now focused to 1 μm^2

$$= \sqrt{\frac{1\,W}{(10^{-6})^2\,m}} = 1.9424 \cdot 10^7$$

so the story is it takes a much stronger field to focus power over a smaller area...