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PSET 2

3.1

a.

The binomial distribution is $p_n(x) = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$

Stirling's approximation is $\ln(n!) \approx n * \ln(n) - n$

From this, we want to derive the Poisson distribution: $p(x) = \frac{e^{-N}N^x}{x!}$

Where $N = np$ and we can assume that n is very large, x is small, and p is small.

Because the Stirling approximation is written in terms of the natural logarithm, I'm going to start by natural logging both sides of the binomial distribution:

$$\ln(p_n(x)) = \ln\left(\frac{n!}{(n-x)!\,x!}p^x(1-p)^{n-x}\right)$$

$$= n\ln(n) - n - (n-x)(\ln(n-x) - n + x) - \ln(x!) + x\ln(p) + (n-x)\ln(1-p)$$

Because p is small, we can use: $\ln(1+x) \approx x$ on the last term. And because x is small, we will drop it out from $n - x$ terms.

Simplifying,

$$= x(\ln(n) + \ln(p)) - pn - \ln(x!)$$

At which point, we'll de-log.

$$p(x) = e^{\ln(np)^x}e^{-pn}e^{\ln(x!)} = (np)^x e^{-pn}\frac{1}{x!} = \frac{e^{-N}N^x}{x!}$$

b.

Next, we want $\langle x(x-1)\ldots(x-m+1)\rangle = N^m$

So we know that $\langle f(x)\rangle = \int f(x)p(x)dx$. Rewriting for discrete space with f as our factorial moments:

$$\langle x(x-1)\ldots(x-m+1)\rangle = \sum_{x=0}^{\infty}\big(x(x-1)\ldots(x-m+1)\big)\frac{e^{-N}N^x}{x!}$$

We also know that $\sum_{x=0}^{\infty}\frac{e^{-N}N^x}{x!} = 1$, so we want a way to rewrite the above sum in that format.

Let's dick around with numbers for a bit: let $m = 5$. Then $(x(x-1)(x-2)(x-3)(x-4))$. If we start summing that, from $x = 0$, a useful thing shows up: for $x < 5$, the product will be zero because one of the sequences will be equal to zero. This means we can shift the bounds of the sum.

$$\sum_{x=0}^{\infty}\big(x(x-1)\ldots(x-m+1)\big)\frac{e^{-N}N^x}{x!} = \sum_{x=m}^{\infty}\big(x(x-1)\ldots(x-m+1)\big)\frac{e^{-N}N^x}{x!}$$

Next, Wikipedia says that a factorial moment is defined as $\big(x(x-1)\ldots(x-m+1)\big) = \frac{x!}{(x-m)!}$

So we have:

$$\sum_{x=m}^{\infty}\left(\frac{x!}{(x-m)!}\right)\frac{e^{-N}N^x}{x!}=\sum_{x=m}^{\infty}\frac{e^{-N}N^x}{(x-m)!}$$

Which looks suspiciously close to the format we want! Based on the denominator, we should make the variable substitution $a = x - m$.

$$\sum_{x=m}^{\infty}\frac{e^{-N}N^x}{(x-m)!}=\sum_{a=0}^{\infty}\frac{e^{-N}N^{a+m}}{(a)!}=\sum_{a=0}^{\infty}\frac{e^{-N}N^aN^m}{(a)!}$$

N^m has no a dependence, so we can pull it out.

$$= N^m\sum_{a=0}^{\infty}\frac{e^{-N}N^a}{(a)!}$$

The sum is now in the same form as before, which we know is normalized.

$$= N^m$$

c.

$$\sigma^2 = \langle (x - \langle x\rangle)^2\rangle$$

$$= \langle x^2 - 2x\langle x\rangle + \langle x\rangle^2\rangle = \langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2 = \langle x^2\rangle - \langle x\rangle^2$$

Of these two, what we don't know is what $\langle x^2\rangle$ is. The derivation in b gives us a hint though.

For $m = 2$, $\langle x(x-1)\rangle = N^2$

Or, $\langle x^2\rangle - \langle x\rangle = N^2$

$$\langle x^2\rangle = N^2 + N$$

So:

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = N^2 + N - N^2 = N$$

$$\sigma = \sqrt{N}$$

$$\frac{\sigma}{\langle x\rangle} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$$

3.2

$$\frac{\sigma}{\langle x\rangle} = \frac{1}{\sqrt{N}}$$

Our previous answer: fractional error goes down as the square root of samples increases. For this, our fractional error is 1% or 0.01.

$$0.01 = \frac{1}{\sqrt{N}}$$

$$10^{-2} = N^{-\frac{1}{2}}$$

$$N = 10^4$$

1 part per million:

$$10^{-6} = N^{-\frac{1}{2}}$$

$$N = 10^{12}$$

Watts = Joules/second

$$E = \frac{hc}{\lambda}$$

For visible light:

$$E = 6.6 * 10^{-34} * 3 * \frac{10^8}{500 * 10^{-9}} = 0.04 * 10^{-17} = 4 * 10^{-19} J$$
$$\rightarrow 4 * 10^{-19} W$$

3.3

a.
20kHz bandwidth
Source impedance: 10kOhm
SNR w/ respect to source Johnson noise is 20dB

$$\langle V_{noise}^2 \rangle = 4kTR\Delta f = 4 * 1.38 * 10^{-23} * 300 * 10 * 10^3 * 20 * 10^3 = 3.3 * 10^{-12}$$
$$SNR = 10 \log_{10} \left(\frac{\langle V_{signal}^2 \rangle}{\langle V_{noise}^2 \rangle} \right)$$
$$20 = 10 \log_{10} \left(\frac{\langle V_{signal}^2 \rangle}{3.3 * 10^{-12}} \right)$$
$$10^2 = \frac{\langle V_{signal}^2 \rangle}{3.3 * 10^{-12}}$$
$$\langle V_{signal}^2 \rangle = 3.3 * 10^{-10}$$
$$V_{signal} = 1.2 * 10^{-5} [V]$$

b.

$$\Delta f = \frac{1}{4RC}$$
$$\langle V_n^2 \rangle = 4kTR \frac{1}{4RC} = \frac{kT}{C}$$
$$10^{-12} = \frac{kT}{C} \rightarrow C = 4.4 * 10^{-9} [F]$$

c.
RMS shot noise 1% of current.

$$\langle I_n^2 \rangle = 2q\langle I \rangle \Delta f$$
$$0.005 * q * 4 * R * C = \langle I \rangle$$
$$\langle I \rangle = 1.4 * 10^{-25} [A]$$
$$\langle I_n^2 \rangle = 64 * 10^{-16} \langle I \rangle$$
$$I_{RMS} = 8 * 10^{-8} \sqrt{\langle I \rangle}$$
$$I_{RMS} = 0.01 \langle I \rangle$$
$$\langle I \rangle = 64 * 10^{-10} [A]$$

3.4

$$x = 0, x = 1$$

$\alpha\, dt$ is $0 \rightarrow 1$ in dt
$\beta\, dt$ is $1 \rightarrow 0$ in dt

We want:

$$\frac{d}{dt} \vec{p} = A * \vec{p}$$

Where $\vec{p} = \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix}$

$p_0(t)$ is the probability of being in state zero and $p_1(t)$ is the probability of being in state one.

Let's reason through an example case, to see how the matrix should look. If we start in state zero, then $p_0(t) = 1$ because we're literally at zero, and $p_1(t) = 0$, because we are not at one. In the next time step, we will either stay at zero or transition to one. Because $\alpha\ dt$ is probability of $0 \to 1$ in dt, this means that $p_1(t+dt) = \alpha dt\ p_0(t)$ for this initial condition. We can do the opposite starting condition to see that $p_0(t+dt) = \beta dt\ p_1(t)$. Our matrix will be 2x2, so we still need to reason out the other two slots. These are occupied the likelihood of staying in the same state, so $1 - \alpha dt$ and $1 - \beta dt$ respectively.

This means that:

$$p_0(t+dt) = 1 - \alpha dt * p_0(t) + \alpha dt * p_1(t)$$
$$p_1(t+dt) = 1 - \beta dt * p_1(t) + \alpha dt * p_0(t)$$

The definition of a derivative is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Forcing the probabilities into this (we'll just do for $p_0(t)$ since it'll look ~the same):

$$\frac{d}{dt} p_0(t) = \lim_{dt \to 0} \left(\frac{\big((1 - \alpha dt) * p_0(t) + \alpha dt * p_1(t) - p_0(t)\big)}{dt} \right)$$

dt drops out for some of the terms, letting us pull out $-\alpha p_0$ and αp_1 from the limit. This ends up nicely leaving behind $\frac{p_0(t) - p_0(t)}{dt}$ in the limit, haha.

$$\frac{d}{dt} p_0(t) = -\alpha p_0(t) + \alpha p_1(t)$$

Analogously,

$$\frac{d}{dt} p_1(t) = -\beta p_1(t) + \beta p_0(t)$$

Putting it in a matrix form:

$$\frac{d}{dt} \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix}$$

And that's part one of six lmao.

Next, we want to solve this by diagonalizing the matrix. Solving this flavor of differential equation using diagonal matrixes leverages that it's easy to solve a differential equation of the form:

$$u' = Du$$

Where D is a diagonal matrix. We can get our equation into this form using a variable substitution. I will admit that the logic for this feels circular and I have to convince myself of the validity frequently.

Our current form is :

$$p' = Ap$$

And what we want to solve is

$$u' = Du$$

If A is diagonalizable, then:

$$p' = PDP^{-1}p$$

The lower case p and uppercase P are collectively a problem, no?

Let's expand the u equation:

$$u' = Du = P^{-1}APu$$

Multiply both sides by P

$$Pu' = APu$$

This suggests that if $Pu' = p'$ is true, then the substitution we want to make is $p = Pu$
We can check through this quickly:

$$P' = Pu' = PDu = PDP^{-1}p = Ap$$

So it we're good!

To diagonalize the matrix, we first will find its eigenvalues.

$$|A - \lambda I| = 0$$

$$\left\| \begin{bmatrix} -\alpha - \lambda & \beta \\ \alpha & -\beta - \lambda \end{bmatrix} \right\| = \alpha\beta + \beta\lambda + \alpha\lambda + \lambda^2 - \alpha\beta = \lambda^2 + \lambda(\alpha + \beta) = \lambda(\lambda + (\alpha + \beta))$$

From this, we have two unique eigenvalues (so the matrix is diagonalizable!), $\lambda = 0$ or $\lambda = -(\alpha + \beta)$.

Let's find some eigenvectors:

$$(A - \lambda_i I)v_i = 0$$

For $\lambda = -(\alpha + \beta)$

$$\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} - \begin{bmatrix} -(\alpha + \beta) & 0 \\ 0 & -(\alpha + \beta) \end{bmatrix} = \begin{bmatrix} \beta & \beta \\ \alpha & \alpha \end{bmatrix}$$

$$\begin{bmatrix} \beta & \beta \\ \alpha & \alpha \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\beta v_1 + \beta v_2 = 0$$

$$\alpha v_1 + \alpha v_2 = 0$$

$$v_1 = -v_2$$

For $\lambda = 0$

$$\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$-\alpha v_1 + \beta v_2 = 0$$

$$\alpha v_1 - \beta v_2 = 0$$

$$v_2 = \frac{\alpha}{\beta} v_1$$

So:

$$\overrightarrow{v_0} = \begin{bmatrix} 1 \\ \frac{\alpha}{\beta} \end{bmatrix}$$

$$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We know that P is made by stacking our eigenvectors with the corresponding entries in D as the eigenvalues. We can manually back this out too, but that's just more effort at this point.

SO:

$$P = \begin{bmatrix} 1 & 1 \\ \alpha/\beta & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & -(\alpha+\beta) \end{bmatrix}$$

Now we shall revisit our mess of u, D, p, P from earlier!

$$u' = \begin{bmatrix} 0 & 0 \\ 0 & -(\alpha+\beta) \end{bmatrix} u$$

This is a recognizable form of ODE, the solution is given by:

$$u = \begin{bmatrix} c_1 \\ c_2 e^{-(\alpha+\beta)t} \end{bmatrix}$$

Now we use our $p = Pu$ relationship:

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \alpha/\beta & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 e^{-(\alpha+\beta)t} \end{bmatrix}$$

This gives us

$$p_0(t) = c_1 + c_2 e^{-(\alpha+\beta)t}$$
$$p_1(t) = \frac{\alpha}{\beta} c_1 - c_2 e^{-(\alpha+\beta)t}$$

c_1, c_2 will be given to us by the initial condition, when $t = 0$.

$$p_0(0) = c_1 + c_2$$
$$p_1(0) = \frac{\alpha}{\beta} c_1 - c_2$$

Solving for c_1, c_2 in terms of $p_0(0)$ and $p_1(0)$: (It's a lot of algebra I don't want to re-type)

$$c_2 = \frac{\alpha p_0(0) - \beta p_1(0)}{\alpha + \beta}$$
$$c_1 = \frac{\beta}{\alpha+\beta}(p_0(0) + p_1(0))$$

Woo.

$$p_0(t) = \frac{\beta}{\alpha+\beta}(p_0(0) + p_1(0)) + \frac{\beta}{\alpha+\beta}(p_0(0) + p_1(0))e^{-(\alpha+\beta)t}$$
$$p_1(t) = \frac{\alpha}{\alpha+\beta}(p_0(0) + p_1(0)) - \frac{\beta}{\alpha+\beta}(p_0(0) + p_1(0))e^{-(\alpha+\beta)t}$$

What a mess. But, $p_0(0), p_1(0)$ can only be some combination of 1 or 0, strictly speaking, so this is cleaner than it looks. In fact, $p_0(0) + \ p_1(0)$ necessarily have to sum to 1, so we can drop them out the above equations.

$$p_0(t) = \frac{\beta}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t}$$
$$p_1(t) = \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t}$$

And there we go, I don't have it in me to separate the factor out.

Unendingly, we are now to find the autocorrelation.

$$\langle x(t)x(t+\tau)\rangle$$

We recall that $\langle f(x)\rangle = \int f(x)p(x)dx$ which in this case will be a sum.

$$\langle x(t)x(t+\tau)\rangle = \sum_x x(t)x(t+\tau)p(t)p(t+\tau)$$

$x(t)$ can only be 1 or 0. $x(t+\tau)$ will also only ever be 1 or 0.
We have the following combinations: 00, 10, 01, 11

Since we're summing over a product, only the 11 combo doesn't result in zero. It results in 1.

$$\langle x(t)x(t+\tau)\rangle = \ p(t)p(t+\tau)$$

Since both states have to be at x=1, we're only looking at p_1 here. We are now asking of the probability for x=1 once we've stepped forward by tau.

$$p_1(t) = \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(t)}$$

$$p_1(t+\tau) = \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(t+\tau)}$$

This gets really messy if we just multiply them.

So let's instead say that the outcome of this should be the same for any t, so let's pick $t = 0$

$$p_1(0) = \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(0)} = \frac{\alpha}{\alpha+\beta}$$

$$p_1(0+\tau) = \frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(0+\tau)}$$
$$p(t)p(t+\tau) = \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)(\tau)}\right)$$

And that's the autocorrelation.

A Lorentzian power spectrum is:

$$S(f) = \frac{2\tau}{1 + (2\pi f\tau)^2}$$

Wiener-Khinchin relates power spectrum to autocorrelation as the inverse FT of it:

$$\langle x(t)x(t+\tau)\rangle = \int_{-\infty}^{\infty} S(f)e^{-i2\pi f\tau}df$$

Or:

$$S(f) = \int_{-\infty}^{\infty} \langle x(t)x(t+\tau)\rangle e^{i2\pi f\tau}d\tau$$

$$= \int_{-\infty}^{\infty} \frac{\alpha}{\alpha+\beta}\left(\frac{\alpha}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}e^{-(\alpha+\beta)(\tau)}\right)e^{i2\pi f\tau}d\tau$$

Well, isn't that nasty.

We know that this is the FT of a delta function. $\int_{-\infty}^{\infty} e^{i2\pi xy}dx = \delta(y)$

$$= \frac{1}{(\alpha+\beta)^2}\int_{-\infty}^{\infty}\left(\alpha^2 e^{-i2\pi f\tau} - \alpha\beta e^{-i2\pi f\tau}e^{-(\alpha+\beta)(\tau)}\right)d\tau$$

The first portion of the integral is a delta function:

$$= \frac{\alpha^2}{(\alpha+\beta)^2}\left(\delta(f) - \frac{\beta}{\alpha}\int_{-\infty}^{\infty}\left(e^{i2\pi f\tau}e^{-(\alpha+\beta)(\tau)}\right)d\tau\right)$$

Evaluating the integral in MATLAB gives:

$$\frac{-e^{\tau(2\pi if-\alpha-\beta)}}{\alpha+\beta-2\pi if}\Bigg|_{-\infty}^{\infty}$$

This doesn't evaluate well or come out to the right format. I have some sense we can do something about the infinities on the basis of the autocorrelation being symmetric (i.e., $a(x) = a(-x) = a(|x|)$) but the form of this is not a Lorentzian, so I'm pausing for now.