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PSET 4

6.1

We are to prove $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ using the summations convention.
I.e., we have:

$$(\vec{B} \times \vec{C})_i = \epsilon_{ijk} B_j C_k$$

Or, with a little switch of indexes:

$$\left(\vec{A} \times (\vec{B} \times \vec{C})\right)_i = \epsilon_{ijk} A_j \left((\vec{B} \times \vec{C})_k\right) = \epsilon_{ijk} A_j \epsilon_{klm} B_l C_m$$

Full disclosure, I do not enjoy the summations convention notation! It is hard to read! It is difficult to remember! Doing it this way saves us a few lines of easy algebra (and a lot of typing) but at what cost, I say? At what cost?

We are given:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Where δ_{ij} is Kroenecker delta. (This is why we picked the above set of indexes).

We are not in matrix land, so we can do a little re-shuffle:

$$\epsilon_{ijk} A_j \epsilon_{klm} B_l C_m = \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m$$
$$= \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m$$
$$= \delta_{il}\delta_{jm} A_j B_l C_m - \delta_{im}\delta_{jl} A_j B_l C_m$$

The δ_{ij} enables us to do a little change of indices:
δ_{il} is one only when $i = l$, so $B_l = B_i$. Similarly $\delta_{jm} = 1$ only when $j = m$, so we can switch $C_m = C_j$. We can do the same for the other half of the equation.

$$= A_j B_i C_j - A_j B_j C_i$$
$$= B_i(A_j C_j) - C_i(A_j B_j)$$
$$= B_i(\vec{A} \cdot \vec{C}) - C_i(\vec{A} \cdot \vec{B})$$

Therefore,

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

For $\nabla = \vec{A} = \vec{B}$ and $\vec{C} = E$, the identity:

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$$

Follows immediately.

6.2

a.

Here we use Gauss' law to find the capacitance between two parallel plates of area A, potential V, and distance D. We ignore boundary effects.

We start by finding E. (I'm neglecting arrows and such for typing convenience.)

Let Q be the charge on each plate. Then the charge density $\rho = \frac{Q}{A}$.

We can write Gauss' law as $\phi_E = \frac{Q}{\epsilon}$ (the electric flux through a surface is proportional to the enclosed charge). You can also do this by writing out the integrals and integrating over a rectangular section across a plate.

Or:

$$EA = \frac{Q}{\epsilon}$$

$$E = \frac{Q}{A\epsilon} = \frac{\rho}{\epsilon}$$

We can then find potential V, which is the work done by the field over distance d, i.e.,

$$V = \frac{\rho}{\epsilon}d$$

Capacitance is then:

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{A\epsilon}d} = \frac{A\epsilon}{d}$$

b.

The internal displacement current will be $\frac{\partial}{\partial t}D$

$$\frac{\partial}{\partial t}D = \frac{\partial}{\partial t}\epsilon E = \frac{\partial}{\partial t}\left(\frac{Q}{A}\right) = \frac{1}{A}\frac{\partial}{\partial t}Q = \frac{I}{A}$$

When we integrate over area A we then get I.

c.

The energy density is given as

$$U = \frac{1}{2}(E \cdot D + B \cdot H)$$

(Excluding vector symbols for my typing convenience.)

We have no magnetic field contributions here so:

$$U = \frac{1}{2}(E \cdot D) = \frac{1}{2}\left(\frac{\rho^2}{\epsilon}\right)$$

In the volume between the plates the energy is then:

$$U_E = \frac{1}{2}\left(\frac{\rho^2}{\epsilon}\right)Ad$$

We are to rearrange this in terms of the capacitance, $C = \frac{A\epsilon}{d}$

We can fully expand out the energy to better see how to substitute it:

$$= \frac{1}{2}\left(\frac{\rho^2}{\epsilon}\right)Ad = \frac{1}{2}\rho * \rho * \frac{1}{\epsilon} * A * d$$

We will substitute in $A = \frac{Cd}{\epsilon}$. Writing it out in a slightly unnatural way:

$$= \frac{1}{2}\frac{\rho}{\epsilon}d\frac{\rho}{\epsilon}d\ C$$

Or

$$U_E = \frac{1}{2}CV^2$$

d.

We have $10[V] * 10[A] = 100[W]$

At 1 hour = 3600 seconds, this is $= 3.6 * 10^5 J$

So we have:

$$3.6 * 10^5 = \frac{1}{2}C(10)^2$$

$$C = 7.2 * 10^3[F]$$

Substituting in to $A = \frac{Cd}{\epsilon}$, we have

$$A = 7.2 * 10^3 * \frac{10^{-6}}{8.85 * 10^{-12}} = 0.81 * 10^9 = 8 * 10^8[m^2]$$

If we are stacking these out of little plates of 10cm side length or $A_{square} = 0.01[m^2]$,

We would require $8 * 10^{10}$ of these stacked. At our given d, this corresponds to a height of $8 * 10^4[m]$.

6.3

a.

Now we play this game for the infinite solenoid with current I and n turns per meter. For the infinite solenoid, the magnetic field will be uniform inside the solenoid, running parallel along its length (directionality determined by direction of current flow). We can reason out that as this solenoid is infinite, the field outside should be zero.

We are given:

$$\oint H\,dl = \int \left(J + \frac{\partial D}{\partial t}\right) dA$$

Parsing this into laymen's terms gives us that the line integral of the magnetic intensity is proportional to the current it encloses (magnetic intensity summed around a closed loop is proportional to the current inside of the loop). We have no $\frac{\partial D}{\partial t}$ in this case, as we have no electric fields.

For some length l of the solenoid, we draw a little box of length l around one side of the solenoid. We will get:

$$Hl = Inl$$

Or:

$$H = nI$$
$$B = \mu nI$$

b.

We have

$$U = \frac{1}{2}(E \cdot D + B \cdot H)$$

This time, with no electric field contributions. So:

$$U = \frac{1}{2}(B \cdot H) = \frac{1}{2}\mu n^2 I^2$$

In a cylindrical volume with radius r and length l this becomes:

$$U_{sol} = \frac{1}{2}\mu(nI)^2 \pi r^2 l$$

c.

We have a big boi magnet of 10T with bore diameter 1m and length 2m. We are told to remember that force is the gradient of potential for a conservative force. Thanks!

We have from the previous question:

$$U_{sol} = \frac{1}{2}\mu(nI)^2 \pi r^2 l$$

So:

$$\frac{\partial U}{\partial r} = \mu(nI)^2 \pi r l$$

We are given the value 10T for B, so we need to re-write this in terms of B:

$$= \frac{B^2}{\mu}\pi r l$$

Plugging in our values with $\mu = 4\pi * 10^{-7}$, we get

$$= 25 * 10^7 [N]$$

6.4
a.

We've got two parallel conductors of infinite length, at 1 meter apart, producing $2 * 10^{-7}$N per meter length force between them. We want to show that 1A is the current that does this. This is terrible sentence structure!

We are given (via Biot-Savart):

$$H = \frac{I}{2\pi r}$$

For the optimal straight conductor.

For a single charge:

$$F = qv \times B$$

We extend this to a line of charge of length L:

$$F = IL \times B$$

Constitutive equation:

$$B = \mu H$$

Putting it all together, (recall the conductors are parallel):

$$F = \frac{I^2 \mu}{2\pi r} L$$

$$\frac{F}{L} = \frac{I^2 \mu}{2\pi r}$$

We plug in our values:

$$\frac{F}{L} = \frac{1 * 4 * \pi * 10^{-7}}{2 * \pi * 1} = 2 * 10^{-7} \left[\frac{N}{m}\right]$$

b.

I assume the problem comes from these "infinite length", "negligible cross-section", and "vacuum" approximations being too approximate. I also think that's a fairly sensitive force measurement and life would be easier if it didn't have to be so precise…

6.5

Well, NIST has published a DIY Lego version of the Kibble balance. Here is the methodology: https://aapt.scitation.org/doi/full/10.1119/1.4929898

a.

Neil's hints are to use the definition of force on an infinitesimal current element:

$$d\vec{F} = Id\vec{l} \times \vec{B}$$

Green's theorem:

$$\int \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) dxdy = \oint (E_x dx + E_y dy)$$

Let's re-write this more or less how Wikipedia gives it to us, so that it looks less like a law about E fields:

$$\oint (Ldx + Mdy) = \iint \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$$

And the formula for the divergence of a magnetic field:

$$\nabla \cdot \vec{B} = 0$$

No monopoles!

We are asked to consider the vertical component of this (we're going to relate it to mg), so let's call this axis z. So our differential z component of the force is:

$$dF_z = I(dl_x B_y - dl_y B_x)$$

We integrate over the path of the coil:

$$F_z = I \int_a^b (dl_x B_y^l - dl_y B_x^l)$$

Where a and b are the end points of the coil, and the superscript l indicates that it's B expressed in the reference coordinates of the coil. Knowing that we want to go to a line integral, we will change our reference frame and take the line integral over the path of the coil:

$$F_z = I \oint (B_y dx - B_x dy)$$

By Green's theorem:

$$I \oint (dxB_y - dyB_x) = I \iint \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) dxdy$$

We will use the divergence now:

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$I \iint \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) dxdy = I \iint \left(\frac{\partial B_z}{\partial z}\right) dxdy$$

We let A be the cross-sectional area of the coil.

$$F_z = I \frac{\partial B_z}{\partial z} A$$

We then have:

$$m = \frac{IA}{g} \frac{\partial B_z}{\partial z}$$

b.

We are given:

$$V(\vec{x},\vec{y}) = -\int_{\vec{x}}^{\vec{y}} \vec{E}\cdot d\vec{l}$$

We revisit the non-two dimensional version of Green's theorem, aka Stokes':

$$\iint \nabla\times\vec{E}\cdot d\vec{A} = \oint \vec{E}\cdot d\vec{l}$$

This in combination with Maxwell's third law:

$$\nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$$

Gives us:

$$V(\vec{x},\vec{y}) = -\iint \frac{\partial\vec{B}}{\partial t}\cdot d\vec{A}$$
$$= -\frac{\partial B_z}{\partial t}A$$

c.

From a we have:

$$m = \frac{IA}{g}\frac{\partial B_z}{\partial z}$$

And from b we have:

$$V = -\frac{\partial B_z}{\partial t}A$$

The Kibble balance exists in part to eliminate the need for making precise measurements of B, so that's the term we want to yeet. We can make good measurement of speed, so we'll substitute that in:

$$V = -\frac{\partial B_z}{\partial t}A = -A\frac{\partial B_z}{\partial z}\frac{\partial z}{\partial t} = -Av\frac{\partial B_z}{\partial z}$$

Now we combine:

$$m = \frac{IV}{vg}$$

d.

The Lego paper says that by doing a "virtual" power measurement, it eliminates the effects of frictional forces (e.g. literal friction, resistance).

6.6

a.

Sunlight has power density 1kW/m^2. Estimate electric field strength associated with this radiation.
We are given:

$$P = E \times H$$

This is the Poynting vector and integrating it over an area gives us the energy being carried by an electromagnetic wave, i.e.:

$$10^3 = \int P * dA$$

We want to re-write the Poynting vector in terms of E.

We are given:

$$\sqrt{\frac{\epsilon_0}{\mu_0}}\hat{k} \times \vec{E} = \vec{H}$$

(Gonna use the arrows for a bit for clarity). So:

$$\vec{P} = \vec{E} \times \sqrt{\frac{\epsilon_0}{\mu_0}}\hat{k} \times \vec{E}$$

The field equation for E is:

$$\vec{E} = \overrightarrow{E_0} e^{i(\vec{k}\,\vec{x}-\omega t)}$$

So we have:

$$\vec{P} = \overrightarrow{E_0} e^{i(\vec{k}\,\vec{x}-\omega t)} \times \sqrt{\frac{\epsilon_0}{\mu_0}}\hat{k} \times \overrightarrow{E_0} e^{i(\vec{k}\,\vec{x}-\omega t)}$$

What we want is the expected value of the magnitude of P. Since the exponential term is a squared sinusoid, its expected value is ½.

$$\langle|\vec{P}|\rangle = \frac{1}{2}E_0^2\sqrt{\frac{\epsilon_0}{\mu_0}}$$

We will do the integral over a 1m^2 area:

$$10^3 = \int \frac{1}{2}E_0^2\sqrt{\frac{\epsilon_0}{\mu_0}} * dA$$

$$10^3 = \frac{1}{2}\sqrt{\frac{8.85 * 10^{-12}}{4 * \pi * 10^{-7}}}E_0^2$$

This gives us $E_0 \approx 868 \left[\frac{V}{m}\right]$

b.

We will do the same thing but for 1W at 1mm square and 1um square.

$$E_0 = \sqrt{\sqrt{\frac{\mu_0}{\epsilon_0}}2W/A}$$

1 square mm:

$$E_0 = \sqrt{377 * 2 * 1 * 1 * 10^6} = 27 * 10^3$$

1 square um:

$$E_0 = \sqrt{377 * 2 * 1 * 1 * 10^{12}} = 27 * 10^6$$