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PSET 6

8.1

The electric field is given by:

$$\vec{E} = \frac{1}{i\omega\mu_0\epsilon_0}\nabla(\nabla\cdot\vec{A}) - i\omega\vec{A}$$

The vector potential is given by:

$$\vec{A}(r) = \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r}\hat{z}$$

$$A_r = \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r}\cos(\theta)$$

$$A_\theta = \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r}\sin(\theta)$$

For spherical coordinates:

$$\nabla\cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

So, this (quite painfully) works out as so (skipping algebra steps because it's quite long):

$$\nabla\cdot\vec{A} = -\frac{\mu_0 I_0 d}{4\pi r^2}e^{-ikr}\cos\theta\,(ikr+1)$$

For spherical coordinates:

$$(\nabla f)_r = \frac{\partial f}{\partial r}$$

$$(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta}$$

We get:

$$(\nabla f)_r = -\frac{\mu_0 I_0}{4\pi}d\cos\theta\left(\frac{e^{-ikr}(-ikr^2-(1+ki)r-k)}{r^3}\right)$$

$$(\nabla f)_\theta = -\frac{\mu_0 I_0}{4\pi r^3}e^{-ikr}(ikr+1)\sin\theta$$

We want to eventually get to:

$$E_\theta = \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{i\omega\mu_0}{r} + \frac{1}{r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{1}{i\omega\epsilon_0 r^3}\right)\sin\theta$$

$$E_r = \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{2}{r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{2}{i\omega\epsilon_0 r^3}\right)\cos\theta$$

We can make a substitution:

$$k^2 = \omega^2/c^2$$
$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

And hypothetically, the algebra is going to work out at this point.

8.2

The Poynting vector describes power flow through a surface. So we draw a sphere at our given distance and calculate the power per area for this geometry. This gives us:

$$\langle P \rangle = \frac{10^3[W]}{4 * \pi * 10^6} = 0.08 * 10^{-3} \left[\frac{W}{m^2}\right]$$

If we flashback to pset 4, we have:

$$\langle |\vec{P}| \rangle = \frac{1}{2} E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}}$$

Which is still valid in this scenario because the wavelength is much, much shorter than the distance we're at.

So we have (for vacuum):

$$0.08 * 10^{-3} = \frac{1}{2} E_0^2 \sqrt{\frac{8.85 \times 10^{-12}}{4\pi \times 10^{-7}}}$$

Solving for E_0^2:

$$E_0 = 0.0077$$

8.3

We are given that the maximum will occur at $R_r = R_l$

$$W = IV = I^2 R_l$$

We can relate this to R_r with the diagram from 8.3:

$$I = \frac{V}{R} = \frac{V}{R_l + R_r}$$

Plugging this back in:

$$W = \left(\frac{V}{R_l + R_r}\right)^2 R_l$$

If we plug in $R_r = R_l$, we will get the relationship from the book. Or, we can take the derivative with respect to one of the Rs.

$$\frac{dW}{dR_l} = V^2((R_l + R_r)^{-2} - 2R_l(R_l + R_r)^{-3})$$
$$0 = (R_l + R_r)^{-2} - 2R_l(R_l + R_r)^{-3}$$

$$0 = R_l + R_r - 2R_l$$
$$R_l = R_r$$

8.4

We are given:

$$\langle P \rangle = \frac{\hat{r} I_0^2 k^2 d^2}{32\pi^2 r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} (\sin\theta)^2$$

The gain is given by:

$$G \equiv \max_{\theta,\varphi} \frac{P(r = 1, \theta, \varphi)}{W/4\pi}$$

Our maximum P is:

$$P_m = \frac{I_0^2 k^2 d^2}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}}$$

So the gain is:

$$G = \frac{I_0^2 k^2 d^2}{8\pi W} \sqrt{\frac{\mu_0}{\epsilon_0}}$$

We are given:

$$W = \frac{I_0^2 \pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d^2}{\lambda^2}$$

Simplifying G:

$$G = \frac{3k^2\lambda^2}{8\pi^2}$$

Further simplifying G (recall $k^2 = \frac{\omega^2}{c^2} = \frac{4\pi^2}{\lambda^2}$):

$$G = \frac{3}{2}$$

Next, we will find A. Again, we must recall from pset 4:

$$\langle |\vec{P}| \rangle = \frac{1}{2} E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}}$$

This times the area will give us the power, which we can also write as:

$$W = \frac{|V|^2}{8R_{load}}$$

Per the text. This means:

$$\frac{|V|^2}{8R_{load}} = \frac{1}{2} E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}} A$$

Or:

$$\frac{|E_0 d|^2}{8R_{load}} = \frac{1}{2} E_0^2 \sqrt{\frac{\epsilon_0}{\mu_0}} A$$

Great, so we can drop the E_0. Now we need to yeet the resistance.
$2W = I_0^2 R_{rad}$ implies that:

$$R = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d^2}{\lambda^2}$$

So:

$$\frac{d^2}{8 \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d^2}{\lambda^2}} = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} A$$

Terrible presentation, but everything but lambda cancels.:

$$A = \frac{3}{8\pi} \lambda^2$$

Our gain:

$$\frac{A}{G} = \frac{\lambda^2}{4\pi}$$