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PSET 7

9.1

We want to derive Snell's law:

$$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

From Fermat's principles, which states that light will take the least-time path.

We can consider one point in a material with index n_1 at a distance a_1 along the x-axis and distance b_1 on the y-axis, sweeping angle θ_1 from the x-axis. Similarly, our second point will be in material with index n_2 at a distance a_2 along the x-axis and distance b_2 on the y-axis, sweeping angle θ_2 from the x-axis.

The total time for a light ray to travel from point 1 to point 2 is:

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{n_1 d_1}{c} + \frac{n_2 d_2}{c} = \frac{1}{c}(n_1 d_1 + n_2 d_2)$$

Where d_1, d_2 is the distance of each point. Re-writing in terms of a and b:

$$t = \frac{1}{c}\left(n_1\sqrt{a_1^2 + b_1^2} + n_2\sqrt{a_2^2 + b_2^2}\right)$$

We can define $a_1 = a_2$ for convenience, and we will re-write b_2 in terms of b_1 so that we can take the derivate with respect to some variable. $b_2 = h - b_1$ where h is the total y-axis difference between point 1 and point 2. I will also drop the subscripts.

$$t = \frac{1}{c}\left(n_1\sqrt{a^2 + b^2} + n_2\sqrt{a^2 + (h-b)^2}\right)$$

$$\frac{\partial t}{\partial b} = \frac{1}{c}\left(n_1 b(a^2 + b^2)^{-\frac{1}{2}} - n_2(h-b)(a^2 + (h-b)^2)^{-\frac{1}{2}}\right)$$

We set this to zero:

$$0 = n_1 b(a^2 + b^2)^{-\frac{1}{2}} - n_2(h-b)(a^2 + (h-b)^2)^{-\frac{1}{2}}$$

Note that the terms:

$$b(a^2 + b^2)^{-\frac{1}{2}}$$

Look like the definition for sine, which is great! So we'll substitute those in:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Which gets us to Snell's law:

$$\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

9.2

a.

Fresnel's equations in the plane of incidence are:

$$E_1 = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} E_0$$

$$E_2 = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0 + \theta_2)\cos(\theta_0 - \theta_2)} E_0$$

And perpendicular to the plane of incidence:

$$E_1 = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0$$

$$E_2 = \frac{2\sin\theta_2 \cos\theta_0}{\sin(\theta_2 + \theta_0)} E_0$$

The Poynting vector is:

$$\vec{P} = \vec{E} \times \vec{H}$$

We do the classic move from the last few psets and re-write the Poynting vector in terms of only E.

$$\vec{H} = \sqrt{\frac{\epsilon}{\mu}} \hat{k} \times \vec{E}$$

Because $\mu_r \approx 1$ in most dielectric materials, we can make the substitution:

$$\vec{H} \approx \frac{1}{\omega\mu_0} \hat{k} \times \vec{E}$$

$$\vec{P} = \frac{n}{\mu c} E^2 \hat{k}$$

So, like in our past psets:

$$|\vec{P}| = \frac{1}{2}\frac{n}{\mu c}|E|^2$$

Which gives that our power ratios are…

For reflected vs incoming:

$$R = \frac{|E_1|^2}{|E_0|^2}$$

And transmitted vs incoming:

$$T = \frac{n_2|E_2|^2}{n_0|E_0|^2}$$

We then plug Fresnel's equations in for the parallel and perpendicular cases, which is quite long and not compelling to type!

b.

In the perpendicular case, we have:

$$R = \left(\frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)}\right)^2$$

But then it's like, what's going on, this is poorly defined at theta=0? Seems rude, doesn't it?

By Snell:

$$\sin(\theta_2) = \sin(\theta_0)\, n_1/n_2$$

We will small-angle approximate this into:

$$\theta_2 = \theta_0 n_1/n_2$$

So then we've got:

$$R = \left(\frac{\sin(\theta_0 n_1/n_2 - \theta_0)}{\sin(\theta_0 n_1/n_2 + \theta_0)}\right)^2$$

Which we'll just small-angle approximate again into:

$$\frac{\left(\frac{\theta_0 n_1}{n_2} - \theta_0\right)^2}{\left(\frac{\theta_0 n_1}{n_2} + \theta_0\right)^2}$$

Then we can bop out the thetas.

$$\frac{\left(\frac{n_1}{n_2} - 1\right)^2}{\left(\frac{n_1}{n_2} + 1\right)^2}$$

Evaluating, we get 0.395, ish.

c.

Brewster's angle is given by:

$$\theta_B = \tan^{-1}\frac{n_2}{n_1}$$

Evaluating w/ 1.5 and 1:

$$\theta_B = 0.98$$

d.

Critical angle, per Wikipedia, is given by:

$$\theta_c = \sin^{-1}\frac{n_2}{n_1}$$

We have to do 1/1.5 (glass to air):

Which is $\theta_c = 0.72$

9.3

It is getting late and I don’t have the energy for this problem, so I will skip it :p such is life!

9.4

9.5

Yeet, boiz, this one looks too long too so I’m skipping it for time :p

9.6

We are given:

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2 x) = -eE(t)$$

This is the damped Lorentz model, where we model an electron as connected via spring-damper (our favorite thing!) to such a large nucleus as to say the nucleus does not move.

We are told this is driven by an oscillator, switching e to q so it doesn’t look like the exponential:

$$m(\ddot{x} + \gamma\dot{x} + \omega_0^2 x) = -qE_0 e^{i(-\omega t)}$$

Neil’s notes cryptically say “susceptibility” and list some equations. With Wikipedia’s help, let’s consider:

$$P = \epsilon_0 \chi_e E$$

Where $\chi_e = \epsilon_r - 1$ and P and E are both vector quantities.

Since our goal to see the complex permittivity, this is how we can introduce the permittivity back in. Here, we refer to Neil’s cryptic notes once more, specifically:

$$p = -ex$$
$$P = Np$$

Or, in my notation:

$$P = -Nqx$$

With this relationship, we can substitute P directly into our differential equation, content in the knowledge that we now have a clear epsilon dependence that can weasel its way in later.

$$-\frac{1}{Nq} m\left(\ddot{P} + \gamma\dot{P} + \omega_0^2 P\right) = -qE_0 e^{i(-\omega t)}$$

$$\ddot{P} + \gamma\dot{P} + \omega_0^2 P = \frac{Nq^2}{m} E_0 e^{-i\omega t}$$

Now we defer to our most favorite trick in the book, which is called don't solve the differential equation by hand. Look up what the form should be. Apparently, we want something like:

$$P(\omega)e^{-i\omega t}$$

Where P is complex.

Let's pop that in.

$$(i\omega)^2 P(\omega)e^{-i\omega t} + \gamma(-i\omega)P(\omega)e^{-i\omega t} + \omega_0^2 P(\omega)e^{-i\omega t} = \frac{Nq^2}{m} E_0 e^{-i\omega t}$$

$$P(\omega) = \frac{\frac{Nq^2}{m} E_0}{-\omega^2 + \omega_0^2 - \gamma i\omega}$$

So now we have an equation for P! Woot!

Let's substitute in our susceptibility based definition in:

$$\epsilon_0(\epsilon_r - 1)E_0 = \frac{\frac{Nq^2}{m} E_0}{-\omega^2 + \omega_0^2 - \gamma i\omega}$$

$$\epsilon_r = 1 + \frac{\frac{Nq^2}{\epsilon_0 m}}{-\omega^2 + \omega_0^2 - \gamma i\omega}$$

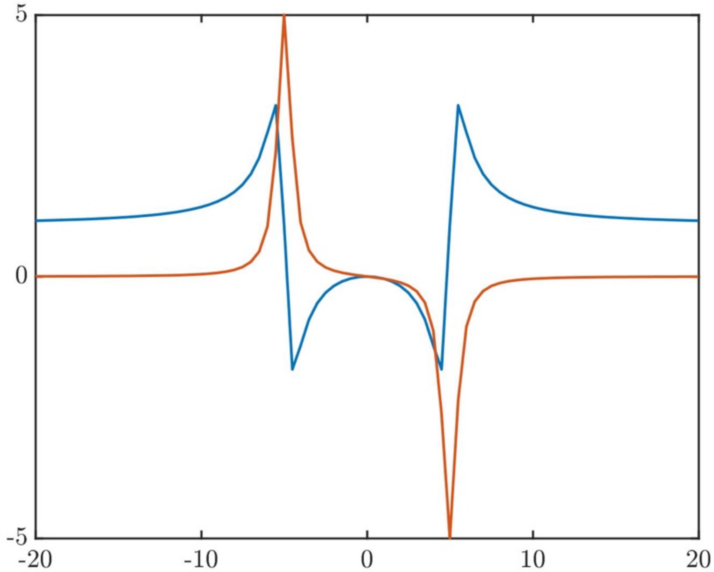
Here we will recall (from what, you may ask) that the plasma frequency is defined as:

$$\omega_p = \sqrt{\frac{n_0 e^2}{\epsilon_0 m}}$$

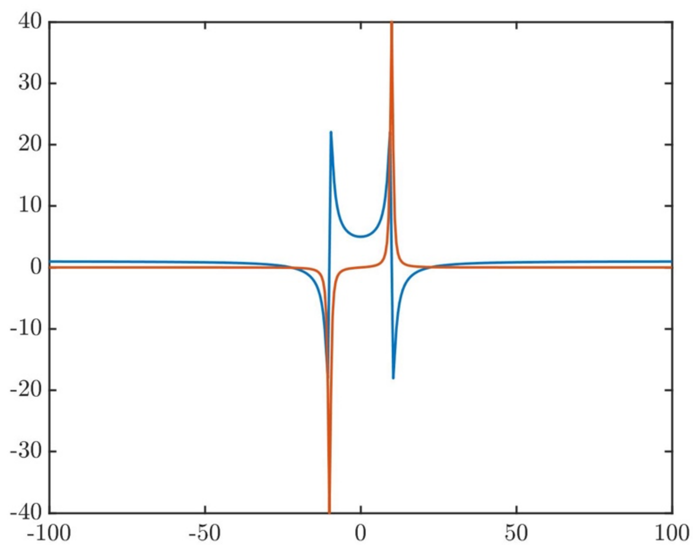
Which in slightly different notation is in here.

$$\epsilon_r = 1 + \frac{\omega_p^2}{-\omega^2 + \omega_0^2 - \gamma i\omega}$$

Here's some graphs with arbitrary values lol.




```
wp = 5;
w0 = 5;
gamma = 1;
```




```
wp = 20;
w0 = 10;
gamma = 1;
```