$\boxed{3.1}$ a) Derive $p(x) = \dfrac{e^{-N} N^x}{x!}$ from binomial dist. + Stirling's approx.

Binomial: $P_n(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $\binom{n}{x} = \dfrac{n!}{(n-x)!\,x!}$

↳ probability of x heads in n trials

Assume n large and p small

↳ Stirling's approx: $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$

$$p(x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

Stirling

$$\approx \frac{\cancel{\sqrt{2\pi}}\, n^{n+\frac{1}{2}} e^{-n}}{\cancel{\sqrt{2\pi}}\,(n-x)^{n-x+\frac{1}{2}} e^{x-n}} \cdot \frac{1}{x!} p^x (1-p)^{n-x}$$

$$= \frac{n^n \sqrt{n}\, \cancel{e^{-n}}}{(n-x)^n (n-x)^{-x} \sqrt{n-x}\, e^x \cancel{e^{-n}}} \cdot \frac{1}{x!} p^x (1-p)^{n-x}$$

$$= \sqrt{\frac{n}{n-x}} \frac{n^n e^{-x}}{(n-x)^{n-x} x!} p^x (1-p)^{n-x}$$

$$\lim_{n\to\infty} p(x) = \frac{n^n e^{-x} p^x (1-p)^{n-x}}{(n-x)^{n-x}\, x!}$$

Set $N = np$ so $p = \frac{N}{n}$

$$= \frac{n^n e^{-x} \left(\frac{N}{n}\right)^x \left(1-\frac{N}{n}\right)^{n-x}}{(n-x)^{n-x}\, x!}$$

factor out n in denominator

$$= \frac{n^n e^{-x} \left(\frac{N}{n}\right)^x \left(1-\frac{N}{n}\right)^{n-x}}{n^{n-x} \left(1-\frac{x}{n}\right)^{n-x} x!}$$

$$= \frac{\cancel{n^n} e^{-x} N^x \cancel{n^{-x}} \left(1-\frac{N}{n}\right)^{n-x}}{\cancel{n^n} \cancel{n^{-x}} \left(1-\frac{x}{n}\right)^{n-x} x!}$$

$$= \frac{e^{-x} N^x \left(1-\frac{N}{n}\right)^{n-x}}{\left(1-\frac{x}{n}\right)^{n-x} x!}$$

For $n \gg x$, $n-x \approx n$

$$\approx \frac{e^{-x} N^x \left(1-\frac{N}{n}\right)^{n}}{\left(1-\frac{x}{n}\right)^{n} x!}$$

For large n, $\left(1-\frac{x}{n}\right)^n \to e^{-x}$

↳ $p(x) \approx \frac{\cancel{e^{-x}}\, N^x e^{-N}}{\cancel{e^{-x}}\, x!}$

$= \boxed{\frac{e^{-N} N^x}{x!}}$

b) $x(x-1)(x-2)\dots(x-m+1) = \frac{x!}{(x-m)!}$

Expected value formula: $E[x] = \sum_x x \cdot p(x)$

$E[X(X-1)(X-2)\dots(X-m+1)]$

$$= \sum_{x=0}^{\infty} (x-1)\dots(x-m+1)\, e^{-N} \frac{N^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x!}}{(x-m)!}\, e^{-N} \frac{N^x}{\cancel{x!}}$$

all terms $x < m$ are zero due to the $(x-m)!$ term

$$= \sum_{x=m}^{\infty} \frac{e^{-N} N^x}{(x-m)!}$$

$$= \sum_{x=m}^{\infty} \frac{e^{-N} N^m N^{x-m}}{(x-m)!}$$

factor out N^m

$$= N^m \sum_{x=m}^{\infty} \frac{e^{-N} N^{x-m}}{(x-m)!}$$

change sum limits

$$= N^m \underbrace{\sum_{x=0}^{\infty} \frac{e^{-N} N^x}{x!}}_{1}$$

b/c this is Poisson dist. and is normalized

$= \boxed{N^m}$

c) For $m=1$, $\frac{x!}{(x-m)!} = x$

$E\left[\frac{x!}{(x-m)!}\right]$ s.t. $m=1$ $= N^1 = N$

$\therefore E[x] = \boxed{N}$

$\sigma = \sqrt{Var(x)}$

$Var(x) = E[x^2] - E[x]^2$

$E[x^2] = E[x(x-1)] + E[x]$

$= E\left[\frac{x!}{(x-2)!}\right] + E[x]$

$$= N^2 + N$$

$$\mathrm{Var}(x) = N^2 + N - N^2$$

$$= N$$

$$\sigma = \sqrt{\mathrm{Var}(x)} = \boxed{\sqrt{N}}$$

$$\therefore \frac{\sigma}{E[x]} = \frac{\sqrt{N}}{N} = \boxed{\frac{1}{\sqrt{N}}}$$

3.2 Avg rate = N photons per sec

Poisson process w/ $\lambda = N$: $\mu = N$, $\sigma = \sqrt{N}$

Relative st. dev. $= \frac{\sigma}{\mu} = \frac{1}{\sqrt{N}}$

$$\frac{1}{\sqrt{N}} \le 0.01 \rightarrow \boxed{N \ge 10{,}000}$$

$$\frac{1}{\sqrt{N}} \le 10^{-6} \rightarrow \boxed{N \ge 10^{12}}$$

Red light: $\lambda = 650\text{ nm}$, $\nu = 4.62 \times 10^{14}\text{ Hz}$, $E = h\nu = 3.06 \times 10^{-19}\text{ J}$

$$N = 10{,}000 \rightarrow \frac{10^4 \cdot 3.06 \times 10^{-19}\text{ J}}{1\text{ s}} = \boxed{3.06 \times 10^{-15}\text{ W}}$$

$$N = 10^{12} \rightarrow \frac{10^{12} \cdot 3.06 \times 10^{-19}\text{ J}}{1\text{ s}} = \boxed{3.06 \times 10^{-7}\text{ W}}$$

3.3 a) $\langle V_{noise} \rangle = \sqrt{4kTR\Delta f}$

RMS voltage

$$= \sqrt{4 \cdot 1.38 \times 10^{-23}\text{ J/K} \cdot 300\text{K} \cdot 10\text{k}\Omega \cdot 20\text{kHz}}$$

$$= 1.82 \times 10^{-6}\ V_{rms}$$

$$SNR = 20 \log_{10}\left(\frac{V_{rms}\text{ signal}}{V_{rms}\text{ noise}}\right)$$

$$20\text{ dB} \le 20 \log_{10}\left(\frac{V_{rms}\text{ signal}}{1.82 \times 10^{-6}\ V_{rms}}\right)$$

$$V_{rms,\text{ signal}} \ge \boxed{1.82 \times 10^{-5}\ V_{rms}}$$

b) Noise bandwidth of RC ckt: $\Delta f = \frac{1}{4RC}$

$$\langle V_{noise} \rangle = \sqrt{4kTR\Delta f}$$

$$1.82 \times 10^{-6}\ V_{rms} = \sqrt{4 \cdot 1.38 \times 10^{-23}\text{ J/K} \cdot 300\text{K} \cdot \cancel{R} \cdot \frac{1}{4\cancel{R}C}}$$

$$\boxed{C = 1.25\text{ nF}}$$

c) 20 kHz bandwidth amp driven by current source I

Size of I for RMS shot noise $= 0.01 \times I$

$\langle I_{noise} \rangle = \sqrt{2q\, I\, \Delta f}$

$0.01\, I = \sqrt{2 \cdot 1.60\times10^{-19}\, C \cdot I \cdot 20\times10^{3}\, Hz}$

$.0001\, I^{\cancel{2}} = \cancel{I} \cdot 6.4\times10^{-15}\, A$

$I = 6.4\times10^{-11}\, A = \boxed{64\ pA}$