

Adiabatic Quantum Computation Is Equivalent to Standard Quantum Computation*

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Abstract. The model of adiabatic quantum computation is a relatively recent model of quantum computation that has attracted attention in the physics and computer science communities. We describe an efficient adiabatic simulation of any given quantum circuit. This implies that the adiabatic computation model and the standard circuit-based quantum computation model are polynomially equivalent. Our result can be extended to the physically realistic setting of particles arranged on a two-dimensional grid with nearest neighbor interactions. The equivalence between the models allows one to state the main open problems in quantum computation using well-studied mathematical objects such as eigenvectors and spectral gaps of Hamiltonians.

Key words. quantum computation, adiabatic computation, nearest neighbor interactions

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I. Introduction. The model of quantum computation has been thoroughly investigated in the last two decades and is by now a well-established one [47]. In this

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model, computation is performed by *quantum circuits*, which are similar to classical (i.e., nonquantum) circuits, except that quantum gates (such as Hadamard and controlled-NOT) are used instead of classical gates (such as OR and NOT).

Around the year 2000, a new model of quantum computation emerged. This model, known as *adiabatic quantum computation*, is based on a celebrated theorem in quantum mechanics known as the *adiabatic theorem* [35, 42] and is quite unlike the usual quantum circuit model. Farhi et al. [25] were the first to study quantum adiabatic algorithms (though related ideas had already appeared more than a decade earlier, e.g., [9, 10, 34]). Farhi et al.'s goal was to attempt to solve hard classical optimization problems such as SATISFIABILITY (SAT). Several simulations (see, e.g., [24]) on random instances of up to 20 quantum bits led to various optimistic speculations. The bad news is that there is now mounting evidence [18, 19, 53] that the algorithm of [25] takes exponential time in the worst case for NP-complete problems. Nevertheless, adiabatic computation was since shown to be promising in other less ambitious directions: it possesses several interesting algorithmic capabilities, as we will soon review, and in addition it exhibits inherent robustness against certain types of quantum errors [17]. We note that small-scale adiabatic algorithms have already been implemented experimentally, using a nuclear magnetic resonance (NMR) system (e.g., [51, 63]).

We briefly describe the model of adiabatic computation (a more precise description appears in section 2.3). A computation in this model is specified by two Hamiltonians named H_{init} and H_{final} (a Hamiltonian is simply a Hermitian matrix). The eigenvector with smallest eigenvalue (also known as the *ground state*) of H_{init} is required to be an easy-to-prepare state, such as a tensor product state. The output of the adiabatic computation is the ground state of the final Hamiltonian H_{final} . Hence, we choose an H_{final} whose ground state represents the solution to our problem. We require the Hamiltonians to be *local*; i.e., we require them to involve only interactions between a constant number of particles (this can be seen as the equivalent of allowing gates operating on a constant number of qubits in the standard model). This, in particular, makes sure that the Hamiltonians have a short classical description, by simply listing the matrix entries of each local term. The running time of the adiabatic computation is determined by the minimal spectral gap¹ of all the Hamiltonians on the straight line connecting H_{init} and H_{final} : $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$ for $s \in [0, 1]$. In particular, the adiabatic computation runs in polynomial time if this minimal spectral gap is at least inverse polynomial.

The motivation for the above definition comes from physics. The Hamiltonian operator corresponds to the energy of the quantum system, and for it to be physically realistic and implementable it must be local. Its ground state is the state of lowest energy. We can set up a quantum system in the ground state of H_{init} (which is supposed to be easy to generate) and apply the Hamiltonian H_{init} to the system. We then slowly modify the Hamiltonian along the straight line from H_{init} towards H_{final} . It follows from the adiabatic theorem that if this transformation is performed slowly enough (how slowly is determined by the minimal spectral gap), the final state of the system will be in the ground state of H_{final} , as required.

What is the computational power of this model? In order to refer to the adiabatic model as a computational model that computes classical functions (rather than quantum states), we consider the result of the adiabatic computation to be the outcome of a measurement of one or more of the qubits, performed on the final ground state. It is

¹The *spectral gap* is the difference between the lowest and second lowest eigenvalues.

known that adiabatic computation can be efficiently simulated by standard quantum computers [18, 24]. Hence, its computational power is not greater than that of standard quantum computers. Several positive results are also known. In [18, 54] it was shown that Grover's quadratic speed-up for an unsorted search [28] can be realized as an adiabatic computation. Moreover, [22, 53, 56] showed that adiabatic computation can "tunnel" through wide energy barriers and thus outperform simulated annealing, a classical counterpart of the adiabatic model. Finally, [7] proved that if we allow nonlocal Hamiltonians, adiabatic quantum computation has the full power of quantum computation.² However, whether or not adiabatic computation with physically realistic Hamiltonians can achieve the full power of quantum computation was not known. In fact, it was not even known whether adiabatic computation can simulate general *classical* computations efficiently. The focus of this paper is the exact characterization of the computational power of adiabatic computation.

Before we describe our results, let us clarify one subtle point. Most of the previous work on the subject focused on a restricted class of adiabatic algorithms known as *adiabatic quantum optimization*. In these algorithms, H_{final} is chosen to be a diagonal matrix, corresponding to a combinatorial optimization problem. In particular, this implies that the ground state of H_{final} (which is the output of the computation) is a classical state, i.e., a state in the computational basis. In this paper, however, we define adiabatic computation as the more general class of adiabatic algorithms in which the only restriction on H_{final} is that it be a local Hamiltonian. We do this because, as noted in [7], from a physical point of view, there is no reason to force the physical process described above to have a diagonal H_{final} , when all other Hamiltonians are not restricted this way. Thus, our definition of adiabatic computation seems to be the natural one to use.³

1.1. Results—Computational Complexity of the Adiabatic Model. Our main result clarifies the question of the computational power of adiabatic algorithms.

THEOREM 1.1. *The model of adiabatic computation is polynomially equivalent to the standard model of quantum computation.*

As mentioned above, one direction of the equivalence is already known [18, 24]. Our contribution is to show that standard quantum computation can be efficiently simulated by adiabatic computation. To prove this we use adiabatic computation with 3-local Hamiltonians.

The search for new quantum algorithms is one of the most important questions in the field of quantum computation. Theorem 1.1 shows that one can instead look for adiabatic quantum algorithms without sacrificing computational power.

There are several reasons why it might be useful to search for quantum algorithms in the adiabatic model. First, it has the benefit of a well-developed physics intuition in the area of adiabatic evolution. In addition, there are powerful mathematical techniques to analyze spectral gaps of matrices in the areas of expander theory [27] and rapidly mixing Markov chains [41, 60]. Indeed, probability theory is

²More precisely, [7] showed that adiabatic computation using *simulatable* Hamiltonians is as powerful as standard quantum computation. Simulatable Hamiltonians are Hamiltonians that can be simulated efficiently by a quantum circuit. They are very different from local Hamiltonians, and cannot even be written explicitly. Instead, such Hamiltonians are specified using products of local unitary matrices.

³Moreover, it seems unlikely that adiabatic quantum optimization can simulate arbitrary quantum computation, since computing the ground state of a classical Hamiltonian is a problem in the complexity class P^{NP} , and it is a long-standing open question whether BQP is contained even in the polynomial time hierarchy (see [11, 38] for definitions).

often used in mathematical physics to analyze spectral gaps of Hamiltonians (see, e.g., [62]). Finally, it is known that many interesting algorithmic problems in quantum computation can be cast as quantum state generation problems [7]. The problem of generating special quantum states seems more natural in the adiabatic model than in the standard model.

1.2. Variants of Theorem 1.1. We mention here a trivial corollary of Theorem 1.1. Local Hamiltonians are a special case of *explicit sparse* Hamiltonians. These are Hermitian matrices that have at most polynomially many nonzero elements in each row and column, and, moreover, for which there is an efficient Turing machine that can generate a list of all nonzero entries in a given row or column. It was shown in [7, 13] that adiabatic quantum computation with explicit sparse Hamiltonians can still be simulated by standard quantum computation. Thus we obtain the following corollary.

COROLLARY 1.2. *The model of adiabatic computation with explicit sparse Hamiltonians is polynomially equivalent to the standard model of quantum computation.*

This corollary might be more useful than Theorem 1.1 in the design of quantum algorithms, due to the extensive mathematical literature on sparse matrices and their spectral gaps. Moreover, it is trivial to generalize it further to allow adiabatic computation with a general path between H_{init} and H_{final} rather than a straight line (see [7, 23] for a rigorous definition).

1.3. Results—Towards Experimental Implications. Theorem 1.1 uses 3-local Hamiltonians that act on qubits (i.e., 2-state particles) that may be arbitrarily far apart. From a practical point of view, it is often difficult to create controlled interactions between particles located far away from each other. Moreover, 3-local Hamiltonians are technologically very difficult to realize. If one wants to physically realize adiabatic algorithms, it would be much better to have only 2-local interactions between nearest neighbor particles. To this end we prove the following theorem.

THEOREM 1.3. *Any quantum computation can be efficiently simulated by an adiabatic computation with 2-local nearest neighbor Hamiltonians operating on 6-state particles set on a two-dimensional grid.*

For recent improvements on this theorem, see subsection 1.6.

Theorems 1.1 and 1.3 open up the possibility of physically realizing universal quantum computation using adiabatically evolving quantum systems. As mentioned before, there is a possible advantage to this approach: adiabatic quantum computation is resilient to certain types of noise [17], mainly due to the existence of a spectral gap in the Hamiltonian. It is well known in physics that such a gap plays an important role in the context of protecting quantum systems from noise (e.g., this is a crucial ingredient in topological and geometrical quantum computation⁴ [32, 39, 50]). However, further study, both experimental and theoretical, is needed to determine the right model for noisy adiabatic computation, and whether fault tolerant adiabatic computation can be achieved (see subsection 1.7).

1.4. Proof of Theorem 1.1: Overview. Given an arbitrary quantum circuit [47], our goal is to design an adiabatic computation whose output is the same as that of

⁴Note, however, that in those models the spectral gap has no effect on the running time or on any other algorithmic aspect, and it is used only to separate the computational subspace from the “noisy” subspace. In contrast, the spectral gap in adiabatic computation is between the correct state of the computation and other states, and is crucial from the algorithmic point of view, since it determines the time complexity of the computation.

the quantum circuit. Some similarities between the models are obvious: one model involves unitary gates on a constant number of qubits, while the other involves local Hamiltonians. However, after some thought, one eventually arrives at the following difficulty. The output state of the adiabatic computation is the ground state of H_{final} . The output state of the quantum circuit is its final state, which is unknown to us. How can we specify H_{final} without knowing the output state of the quantum circuit? Notice that this state can be some complicated quantum superposition. One might wonder why our task is not trivial, since this state does have an efficient local classical description, namely, the quantum circuit. However, local quantum gates, which operate in sequence to generate a nonlocal overall action, are very different from local Hamiltonians, which correspond to simultaneous local constraints. To explain the solution, we first set some notation.

Without loss of generality we assume that the input to the quantum circuit consists of n qubits all initialized to $|0\rangle$'s.⁵ Then a sequence of L unitary gates, U_1, \dots, U_L , each operating on one or two qubits, is applied to the state. The system's state after the ℓ th gate is $|\alpha(\ell)\rangle$. The output of the quantum circuit is in general a complicated quantum state $|\alpha(L)\rangle$ of n qubits, which is then measured in the standard basis. We now want to associate with it a corresponding adiabatic computation.

A first natural attempt would be to define H_{final} as a local Hamiltonian with $|\alpha(L)\rangle$ as its ground state. However, this attempt encounters the difficulty mentioned above: not knowing $|\alpha(L)\rangle$, it seems impossible to explicitly specify H_{final} .

The key to resolving this difficulty is a beautiful idea that goes back to Feynman [26], which we call the circuit-to-Hamiltonian construction. This idea was a core ingredient in an ingenious result of Kitaev [38], which inspired our work. In this result Kitaev provides the first complete problem for the class QMA, the quantum analogue of NP. Roughly speaking, QMA is defined like NP, except the verifier is a quantum algorithm and the witness is a quantum state. Kitaev shows that for some constant k (5, to be precise), the k -local Hamiltonian problem is QMA-complete. This problem is defined roughly as follows. The input is a local Hamiltonian on n qubits, of polynomially many terms involving k qubits each. The goal is to estimate the ground state energy of the Hamiltonian to within inverse polynomial precision. The exact definition of QMA and the local Hamiltonian problem are not important for us (see [38]). What we need presently from Kitaev's result is the circuit-to-Hamiltonian construction on which it is based. We now explain how it works.

To understand the idea, it is best to first consider the celebrated classical Cook–Levin theorem [11], which Kitaev's result generalizes to the quantum case. The Cook–Levin theorem states that for some constant k (3, to be precise), k -SAT is NP-complete. The idea of the proof is that correct classical computation can be verified locally. In slightly more detail, we consider the history of the computation performed by a verifier of the NP language. The computation starts with an input x and a witness y written on the tape. One can write the state of the tape at time 0 as the first row in a two-dimensional tableau. Then the state of the tape at the next time step can be written in the next row, and so on. The point is that since the computation is performed by a local Turing machine which evolves by modifying only a constant number of places on the tape in each time step, it is possible to check that a given tableau is a valid history of an accepting computation, using local tests, namely, by considering windows of constant size in the tableau. These local constraints form a k -SAT formula which is satisfiable if and only if there exists a valid accepting tableau

⁵Otherwise, the first n gates can be used to flip the qubits to the desired input.

for the input x , which is equivalent to saying that the input x is accepted by the NP verifier.

For his proof of the quantum analogue of this theorem, Kitaev needed to define a local Hamiltonian that checks the correct propagation of the quantum circuit corresponding to the verifier of the QMA language. Following Feynman, Kitaev [38] constructed a Hamiltonian whose ground state is the entire *history* of the quantum computation of the verifier, in *superposition*:

$$(1) \quad |\eta\rangle := \frac{1}{\sqrt{L+1}} \sum_{\ell=0}^L |\alpha(\ell)\rangle \otimes |1^\ell 0^{L-\ell}\rangle^c.$$

The right (L qubit) register is a clock that counts the steps by adding 1's from left to right. The superscript c denotes clock qubits. The idea is that the unary representation of the clock enables a local verification of correct propagation of the computation from one computational step to the next. To explain why Kitaev needs a clock, and does not use a tableau of quantum states, consider a computational step in which no computation is performed. The two consecutive rows in the tableau need to be the same in this case. In the classical case, the fact that two rows are the same can of course be checked location by location. In the quantum case, on the other hand, two given states can be very different even if all their local reduced density matrices are the same, and so we cannot verify that the states are equal by comparing their local reduced density matrices one by one! The entanglement with the clock is what enables the checks to be local nevertheless. We refer the reader to [6] for more on the analogy between the classical and quantum Cook–Levin theorems. Notice that this construction of a local Hamiltonian whose ground state is the history state $|\eta\rangle$ of a given quantum circuit can be done for any quantum circuit and not only to QMA verifiers.

Our key idea is to use the circuit-to-Hamiltonian construction for the circuit we would like to simulate adiabatically, and take the resulting Hamiltonian to be the final Hamiltonian H_{final} of the adiabatic computation. The ground state of this Hamiltonian is the history state of the circuit we are trying to simulate, namely, $|\eta\rangle$. Extracting the output of the quantum circuit from the history state is easy: Measure all the qubits of the clock and if the clock is in the state $|1^L\rangle$, the computational qubits carry the output of the circuit. Otherwise, start from scratch.⁶

For the initial Hamiltonian H_{init} we require that it have $|\alpha(0)\rangle \otimes |0^L\rangle^c$, the first term in the history state, as its unique ground state. It is easy to define such a local Hamiltonian, because $|\alpha(0)\rangle \otimes |0^L\rangle^c$ is a tensor product state. Crucially, H_{init} and H_{final} can be constructed efficiently from the given quantum circuit; no knowledge of $|\alpha(L)\rangle$ is required for the construction.

A technical problem lies in showing that the spectral gap of the intermediate Hamiltonians $H(s)$ is lower-bounded by some inverse polynomial (more specifically, we show it is larger than $1/L^2$). To do this, we notice that the state in the adiabatic evolution evolves in some low dimension invariant subspace, spanned by the states of the circuit at various times. It thus suffices to consider the Hamiltonian in this subspace. We map the restricted Hamiltonian to a Markov chain corresponding to a random walk on the $L+1$ time steps. We then apply the conductance bound from the theory of rapidly mixing Markov chains [60] to lower bound the spectral gap of this chain by $\Omega(1/L^2)$. This completes the proof of Theorem 1.1.

⁶This gives an overhead factor of L . This factor can be avoided; see the end of this subsection, and subsection 3.3.

The proof outlined above does not provide a lower bound on the *global spectral gap*, i.e., on the spectral gap inside the entire Hilbert space. Instead, we used the fact that the adiabatic evolution is confined to some small invariant subspace and analyzed the spectral gap of the Hamiltonian restricted to that subspace. This does not imply anything about the global spectral gap, since there may be states of arbitrary energy outside of the invariant subspace. Such a situation might be problematic for experimental implementation, in which noise might induce transitions to outside of the invariant subspace. In particular, we refer to constructions in which the global spectral gap is 0 (i.e., the ground state is not unique) as *degenerate adiabatic computations*. As we shall show in Lemma 4.1, our adiabatic computation does have a nonnegligible global spectral gap, which makes it possibly more relevant for physical implementations and will also be used in our later results.

The next step we make is to improve the construction from 5-local to 3-locality. The proof is based on a simple idea (used in [36] to prove that the 3-local Hamiltonian problem is QMA-complete). Here, there is no longer an invariant subspace. Obtaining a lower bound on the (now inevitably global) spectral gap requires some additional technical issues beyond those that we used to prove the global spectral gap for the 5-local case.

One final point worth mentioning is how to avoid the factor L increase in the running time as a result of ending with the history state instead of the final state of the circuit. One easy way to do this is to add to the end of the quantum circuit being simulated $O(\frac{1}{\epsilon}L)$ identity gates. This has the effect that most of the history state $|\eta\rangle$ is concentrated on the final state $|\alpha(L)\rangle$. See subsection 3.3 for more details.

Following this work, Siu [61] suggested an alternative and somewhat more efficient way to avoid the extra L factor, which also has the advantage that the adiabatic computation ends in the desired final state of the circuit rather than in the history state. His construction, however, necessarily gives degenerate adiabatic computations. Siu's Hamiltonian path consists of two parts. The first part is identical to ours, leading us from the initial state $|\alpha(0)\rangle \otimes |0^L\rangle^c$ to the history state $|\eta\rangle$. For the second part, one observes that by applying (a slight modification of) our construction to the inverted circuit $U_L^\dagger, \dots, U_1^\dagger$, one obtains an adiabatic evolution taking the state $|\alpha(L)\rangle \otimes |1^L\rangle^c$ to the history state. In the second part of his construction, Siu applied the time reversal of this latter evolution, taking $|\eta\rangle$ to $|\alpha(L)\rangle \otimes |1^L\rangle^c$.

1.5. Proof of Theorem 1.3: Overview. The idea underlying the proof of Theorem 1.1 by itself does not suffice to prove Theorem 1.3. The basic problem lies in arranging sufficient interaction between the computational and clock particles, since if the particles are set on a grid, each clock particle can interact with only four neighbors. We circumvent this problem as follows. Instead of having separate clock and computational particles, we now assign to each particle both clock and computational degrees of freedom (this is what makes our particles 6-state). We then construct a computation that propagates locally over the entire set of particles, snaking up and down each column of the lattice. The adiabatic evolution now ends up in the history state of this snake-like sequence of states.

The lower bound on the spectral gap is obtained in an essentially identical way to the 3-local Hamiltonian case.

1.6. Discussion of Follow-up Work. Since the initial publication of our results in 2004 [3], the study of adiabatic computation and computational complexity of quantum Hamiltonian systems in general has advanced considerably, partly motivated by our work. We now summarize some of those recent developments.

Improvements in Parameters. Motivated mainly by experimental considerations, much of the follow-up work focused on improving the various parameters in Theorems 1.1 and 1.3, such as the running time, the number of states of the particles, and their geometry.

First, [37] improved Theorem 1.1 and showed that 2-local interactions (as opposed to our 3-local interactions) between qubits (i.e., 2-state particles) are sufficient to achieve adiabatic universality. The interactions they use are not confined to a grid but occur between any two qubits. Oliveira and Terhal [48] improved the result further and showed how to achieve adiabatic universality with a two-dimensional grid of qubits with nearest neighbor interactions. Other improvements can be found in Nagaj's thesis [44]. A somewhat related construction of adiabatic computation with local Hamiltonians was suggested by Mizel, Lidar, and Mitchell [43]; they sketch a proof of correctness, which presumably can be extended into a complete proof.

For a while, it was speculated that one-dimensional systems cannot achieve universality. Somewhat surprisingly, Aharonov et al. [4] recently showed that one-dimensional quantum systems of 9-state particles with nearest neighbor interactions *can* achieve adiabatic universality using techniques similar to those in the proof of Theorem 1.3. Recently, Pepper [52] managed to improve this to 7-state particles; it is an open question whether one can get all the way down to qubits.

The results in this paper are far from optimal in terms of the running time of the adiabatic algorithm. In [20], Deift, Ruskai, and Spitzer provided an alternative and somewhat simpler analysis of the Hamiltonians we describe here, which in particular improved the bound on the spectral gap in Lemma 4.1 by a factor of L , and might translate to better bounds on the running time. Further improvements were achieved by Nagaj [44].

Connection with QMA-Completeness. Prior to our work, circuit-to-Hamiltonian constructions (as in [38] and follow-up work) were primarily used for proving that certain local Hamiltonian problems are QMA-complete. One of our main contributions is the realization that the circuit-to-Hamiltonian method can also be used to prove the universality of adiabatic computation. Thanks to this realization, most circuit-to-Hamiltonian constructions nowadays imply two separate results: QMA-completeness and adiabatic universality. This is, for instance, the case in much of the follow-up work mentioned above, such as [37], [48], and [4].

One should note, however, that there is currently no general statement showing how to obtain the two results from a circuit-to-Hamiltonian construction. Instead, these results are obtained on a case-by-case basis. Roughly speaking, proving QMA-completeness is easier than proving adiabatic universality since the latter requires bounding the minimal spectral gap along the entire Hamiltonian path, whereas the former requires arguing about the ground energy of just one Hamiltonian (indeed, some works, such as [15, 21, 45], prove only a QMA-completeness result, even though presumably one can extend them to adiabatic universality). On the other hand, adiabatic universality is sometimes easier to achieve if we allow degenerate adiabatic computation, since in this case it suffices to analyze the spectral gap in an invariant subspace. The local Hamiltonian problem, by definition, applies to the entire Hilbert space and cannot be restricted to a subspace. Such a case, for instance, showed up in [4]: their QMA-completeness result required 12-state particles, whereas their (degenerate) adiabatic universality result required only 9-state particles.

Circuit-to-Hamiltonian constructions similar to the one presented in the proof of Theorem 1.3 were used also outside of the context of adiabatic computation or QMA-completeness; see, e.g., [46, 58].

Fault Tolerance. The fact that adiabatic computation is universal for quantum computation motivates studying the possibility of experimentally realizing quantum computation using adiabatic systems. Since noise is inevitable in any physical system, a major question is whether noisy adiabatic systems (with a realistic model of noise) can be made as powerful (from a computational point of view) as ideal adiabatic computers. In other words, we ask whether adiabatic systems can be made fault tolerant. Fault tolerance for the standard quantum circuit model was shown already in 1996 (see [2] and the references therein).

Several researchers have begun to study noisy adiabatic computation, and various methods to protect the computation against noise were suggested (see, e.g., [1, 40, 55, 57]). In particular, quantum error correcting codes tailored for adiabatic evolution were discovered by Jordan, Farhi, and Shor [33], and it was shown that these codes achieve significant robustness to noise. They quantify noise in terms of the temperature in the system, and argue that whereas without error correcting, the temperature must be inverse polynomial, when error corrections are applied, it suffices that the temperature behaves inverse logarithmically in the size of the computation. We believe that more needs to be done in this direction; see the following section for open questions.

1.7. Open Questions. This paper demonstrates that quantum computation can be studied and implemented entirely within the adiabatic computation model, without losing any computational power. This result raises several interesting open questions.

Obviously, an important problem is to minimize the physical requirements for adiabatic universality as much as possible. In particular, can adiabatic universality be achieved with nearest neighbor interacting qubits on a line? This might have experimental implications.

One of the major motivations for this direction of research is the possibility that adiabatic computation, due to its inherent robustness to certain types of noise, might be better suited for implementation than the standard circuit-based model. We believe that whether this is really the case is still not clear, and depends in part on whether fault tolerance can be achieved in this model, and on whether algorithms can be efficiently implemented in this model.

A related interesting open problem is determining how large the spectral gap can be in universal adiabatic systems. A large spectral gap is expected to improve the robustness to thermal noise. It seems that the answer to this question might depend on the geometry of the system. In [49], Osborne proved that adiabatic evolutions of one-dimensional systems can be simulated efficiently for a short time if the Hamiltonian has a constant energy spectral gap (assuming that the spectral norm of the Hamiltonian is also bounded above by a constant, as otherwise this statement is vacuous). Using area law results, Hastings [29] improved these simulations to an arbitrary amount of time. Thus, unless quantum computation can be efficiently simulated by classical computation (which is considered unlikely), we cannot hope to prove universality of adiabatic evolution in one dimension using Hamiltonians with constant spectral gap. It is an open question whether this can be achieved using particles on a higher dimensional grid, or even using particles with no particular geometry (i.e., allowing interaction between any k particles). Such a result would imply an adiabatic analogue of the threshold theorem for the standard model of quantum computation.

This question seems closely related to the question of finding quantum analogues of the celebrated PCP theorem (see [11]), one of the most important results in theoretical computer science. This possibility is discussed further in [4].

Finally, one other motivation for this work is our hope that the adiabatic framework might lead to the discovery of new quantum algorithms. As shown in this paper, as well as in [7], tools from probability theory, mathematical physics, and spectral gap analysis might turn out to be relevant and useful. In order to improve our understanding of the adiabatic paradigm, it might be insightful to see adiabatic versions of known quantum algorithms, presented in a meaningful way (of course, by now we know that adiabatic versions exist, but the question is whether the adiabatic point of view can add any insight). It might be useful in this context to clarify the connection between adiabatic algorithms and quantum walk algorithms which are also known to be universal for quantum computation (see [16] and the references therein). The two seem to be tightly related in the case of adiabatic computations resulting from circuit-to-Hamiltonian constructions.

Organization. In section 2 we describe the model of adiabatic computation and state some relevant facts about Markov chains. In section 3 we prove Theorem 1.1 by showing how adiabatic systems with 5-local Hamiltonians can efficiently simulate standard quantum computations. Section 4 proves the global spectral gap and improves the construction to one using only 3-local Hamiltonians. In section 5 we prove Theorem 1.3 by showing how to adapt the construction to a two-dimensional grid.

2. Preliminaries.

2.1. Hamiltonians of n -Particle Systems. For background on n -qubit systems, quantum circuits, and Hamiltonians, see [47]. A system consisting of n d -state particles is described by a state in Hilbert space of dimension d^n , the tensor product of n d -dimensional Hilbert spaces. For simplicity, we restrict our discussion in this subsection to quantum systems composed of 2-state particles, i.e., qubits; a similar discussion holds for higher dimensional particles (such as the 6-state particles we consider later).

In the standard model of quantum computation, the state of n qubits evolves in discrete time steps by unitary operations. In fact, the underlying physical description of this evolution is continuous, and is governed by Schrödinger's equation: $-i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$. Here $|\psi(t)\rangle$ is the state of the n qubits at time t , and $H(t)$ is a Hermitian $2^n \times 2^n$ matrix operating on the space of n qubits. This $H(t)$ is the *Hamiltonian* operating on a system; it governs the dynamics of the system. Given that the state of the system at time $t = 0$ is equal to $|\psi(0)\rangle$, one can in principle solve Schrödinger's equation with this initial condition to get $|\psi(T)\rangle$, the state of the system at a later time $t = T$. The fact that the Hamiltonian is Hermitian corresponds to the familiar fact that the discrete time evolution of the quantum state from time t_1 to a later time t_2 is unitary.

We sometimes refer to eigenvalues of Hamiltonians as *energies*. The *ground energy* of a Hamiltonian is its lowest eigenvalue and the corresponding eigenvector(s) are called *ground state(s)*. We define $\Delta(H)$, the *spectral gap* of a Hamiltonian H , to be the difference between the lowest eigenvalue of H and its second lowest eigenvalue. ($\Delta(H) = 0$ if the lowest eigenvalue is degenerate, that is, has more than one eigenvector associated with it.) We define the *restriction* of H to some subspace \mathcal{S} , denoted $H_{\mathcal{S}}$, as $\Pi_{\mathcal{S}} H \Pi_{\mathcal{S}}$, where $\Pi_{\mathcal{S}}$ is the orthogonal projection on \mathcal{S} .

A Hamiltonian on an n -particle system represents a certain physical operation that one can, in principle, apply to an n -particle system. However, it is clear that one

cannot efficiently apply any arbitrary Hamiltonian (even describing a Hamiltonian on n qubits requires exponential space in the worst case). We say that a Hamiltonian H is k -local if H can be written as $\sum_A H^A$, where A runs over all subsets of k particles, and H^A operates trivially on all but the particles in A (i.e., it is a tensor product of a Hamiltonian on A with identity on the particles outside of A). Notice that for any constant k , a k -local Hamiltonian on n -qubits can be described by $2^{2k}n^k = \text{poly}(n)$ numbers. We say that H is local if H is k -local for some constant k .

In this paper we restrict our attention to k -local Hamiltonians. This requirement corresponds to the fact that all known interactions in nature involve a constant number of particles. We attempt to make k as small as possible to make the Hamiltonian presumably easier to implement.

2.2. The Adiabatic Theorem. The cornerstone of the adiabatic model of computation is the celebrated adiabatic theorem [35, 42]. Consider a time-dependent Hamiltonian $H(s)$, $s \in [0, 1]$, and a system initialized at time $t = 0$ in the ground state of $H(0)$ (here and in the following we assume that for all $s \in [0, 1]$, $H(s)$ has a unique ground state). Let the system evolve according to the Hamiltonian $H(t/T)$ from time $t = 0$ to time T . We refer to such a process as an *adiabatic evolution according to H for time T* . The adiabatic theorem affirms that for large enough T the final state of the system is very close to the ground state of $H(1)$. Just how large T should be for this to happen is determined by the spectral gap of the Hamiltonians $H(s)$ and their norm. Such an upper bound on T is given in the following theorem, adapted from a paper by Jansen, Ruskai, and Seiler [31]. See also [8] for an elementary proof of a slightly weaker version.

THEOREM 2.1 (the adiabatic theorem, adapted from [31, Theorem 3]). *Let H_{init} and H_{final} be two Hamiltonians acting on a quantum system and consider the time-dependent Hamiltonian $H(s) := (1 - s)H_{\text{init}} + sH_{\text{final}}$. Assume that for all s , $H(s)$ has a unique ground state, and that*

$$(2) \quad T \geq \Omega \left(\frac{\|H_{\text{final}} - H_{\text{init}}\|^2}{\epsilon \min_{s \in [0, 1]} \{\Delta^3(H(s))\}} \right)$$

for some $\epsilon > 0$. Let $|\psi(T)\rangle$ be the solution at time $t = T$ for Schrödinger's equation $-i \frac{d}{dt} |\psi(t)\rangle = H(t/T) |\psi(t)\rangle$ with the initial state $|\psi(0)\rangle$ being the ground state of H_{init} . Then $|\psi(T)\rangle$ is ϵ -close in ℓ_2 -norm to the ground state of H_{final} (with an appropriate setting of global phase). The matrix norm is the spectral norm $\|H\| := \max_{w \neq 0} \|Hw\|_2 / \|w\|_2$.

There have been several claims in the literature that the above dependence on the spectral gap Δ can be improved to quadratic instead of cubic (and in earlier versions of our paper we used such a statement). Unfortunately, it seems that none of these claims is backed by a rigorous proof (and in some cases mistakes were found in such purported proofs). For more details, we refer the reader to the discussion in [31].

Still, in many cases, the bound on T in the above theorem *can* be significantly improved. This is the case when the spectral gap is close to its minimum only for a very short interval of s , outside of which the spectral gap is much larger. In such cases, one can apply Theorem 3 of [31] in its original and more precise form in order to obtain better bounds on T . An example of this is shown in section VI of [31]. In this paper we make no such attempts to optimize our bounds on T .

2.3. The Model of Adiabatic Computation. Let us now describe the model of adiabatic computation. In this paper we use the following definition of adiabatic com-

putation that slightly generalizes that of Farhi et al. [25]. An adiabatic computation is specified by H_{init} and H_{final} . Its output is (close to) the ground state of H_{final} .

DEFINITION 2.2. *A k -local adiabatic computation $AC(n, d, H_{\text{init}}, H_{\text{final}}, \epsilon)$ is specified by two k -local Hamiltonians, H_{init} and H_{final} , acting on n d -state particles, such that both Hamiltonians have unique ground states. The ground state of H_{init} is a tensor product state. The output is a state that is ϵ -close in ℓ_2 -norm to the ground state of H_{final} . Let T be the smallest time such that the final state of an adiabatic evolution according to $H(s) := (1-s)H_{\text{init}} + sH_{\text{final}}$ for time T is ϵ -close in ℓ_2 -norm to the ground state of H_{final} . The running time of the adiabatic algorithm is defined to be $T \cdot \max_s \|H(s)\|$.*

Observe that we have chosen our definition of running time to be $T \cdot \max_s \|H(s)\|$ and not T . To see why, notice that according to Schrödinger's equation, for any $c > 0$, the final state of a system that evolves according to some Hamiltonian $H(s)$ for time T is identical to that of a system that evolves according to $cH(s)$ for time T/c . In fact, this is a basic physical trade-off between energy and time. This trade-off can also be seen in Theorem 2.1: if both H_{init} and H_{final} are multiplied by some factor $c > 0$, the resulting bound on T gets divided by the same factor. Hence, one can achieve an arbitrarily small value of T by multiplying the Hamiltonians by some large factor. This clearly shows that T is not a meaningful notion of running time. On the other hand, our notion of running time is invariant under scaling.

The right-hand side of (2) can be used to provide an upper bound on the running time of an adiabatic computation. Hence, in order to show that an adiabatic algorithm is efficient, it is enough to use Hamiltonians of at most $\text{poly}(n)$ norm, and show that for all $s \in [0, 1]$ the spectral gap $\Delta(H(s))$ is at least inverse polynomial in n .

We note that in certain cases, it is possible to obtain a stronger upper bound on the running time. Indeed, assume there exists a subspace \mathcal{S} such that for all $s \in [0, 1]$, $H(s)$ leaves \mathcal{S} invariant, i.e., $H(s)(\mathcal{S}) \subseteq \mathcal{S}$. Equivalently, $H(s)$ is block diagonal in \mathcal{S} and its orthogonal space \mathcal{S}^\perp . Consider $H_{\mathcal{S}}(s)$, the restriction of $H(s)$ to \mathcal{S} . Then, starting from a state inside \mathcal{S} , an adiabatic evolution according to H is *identical* to an adiabatic evolution according to $H_{\mathcal{S}}$ (this follows from Schrödinger's equation). Hence, we can potentially obtain a stronger upper bound by replacing $\Delta(H(s))$ with $\Delta(H_{\mathcal{S}}(s))$ in (2). This observation will be used in section 3.

Finally, as mentioned in subsection 1.2, there are other, more general definitions of adiabatic computation (allowing for more general Hamiltonians, or general paths). By considering a restricted model (of local Hamiltonians with a straight path) we are obviously only making our result stronger.

2.4. Markov Chains and Hermitian Matrices. Under certain conditions, there exists a standard mapping of Hamiltonians to Markov chains (for background on Markov chains, see [41]). The following fact is useful to show that this mapping applies in the case we analyze.

FACT 2.3 (adapted from Perron's theorem, Theorem 8.2.11 in [30]). *Let G be a Hermitian matrix with real nonnegative entries. If there exists a finite k such that all entries of G^k are positive, then G 's largest eigenvalue is positive, and all other eigenvalues are strictly smaller in absolute value. Moreover, the corresponding eigenvector is unique, and all its entries are positive.*

We define the mapping for G , a Hermitian matrix operating on an $L + 1$ -dimensional Hilbert space. Suppose that all the entries of G are real and nonnegative, that its eigenvector $(\alpha_0, \dots, \alpha_L)$ with largest eigenvalue μ satisfies $\alpha_i > 0$ for all $0 \leq i \leq L$,

and that $\mu > 0$. Define P by

$$(3) \quad P_{ij} := \frac{\alpha_j}{\mu\alpha_i} G_{ij}.$$

The matrix P is well defined, and it is stochastic because all its entries are nonnegative and each of its rows sums up to one. It is easy to verify the following fact.

FACT 2.4. *The vector (v_0, \dots, v_L) is an eigenvector of G with eigenvalue δ if and only if $(\alpha_0 v_0, \dots, \alpha_L v_L)$ is a left eigenvector of P with eigenvalue δ/μ .*

We will consider G of the form $G = I - H$ for some Hamiltonian H . The above fact implies that if $(\alpha_0, \dots, \alpha_L)$ is the ground state of H with eigenvalue λ , then $(\alpha_0^2, \dots, \alpha_L^2)$ is a left eigenvector of P with maximal eigenvalue 1. By normalizing, we obtain that $\pi := (\alpha_0^2/Z, \dots, \alpha_L^2/Z)$ is the limiting distribution of P , where $Z = \sum \alpha_i^2$. Moreover, the gap between P 's largest and second largest eigenvalues is equal to $\Delta(H)/(1 - \lambda)$.

2.5. Spectral Gaps of Markov Chains. Given a stochastic matrix P with limiting distribution π , and a subset $B \subseteq \{0, \dots, L\}$, the flow from B is given by $F(B) := \sum_{i \in B, j \notin B} \pi_i P_{ij}$. Define the π -weight of B as $\pi(B) := \sum_{i \in B} \pi_i$. The conductance of P is defined by $\varphi(P) := \min_B F(B)/\pi(B)$, where we minimize over all nonempty subsets $B \subseteq \{0, \dots, L\}$ with $\pi(B) \leq \frac{1}{2}$.

THEOREM 2.5 (the conductance bound [60]). *The eigenvalue gap of P is at least $\frac{1}{2}\varphi(P)^2$.*

3. Equivalence of Adiabatic and Quantum Computation. Here we prove Theorem 1.1 by showing how to simulate a quantum circuit consisting of L 2-qubit gates on n qubits by an adiabatic computation on $n + L$ qubits (the other direction was shown in [18, 25]). We allow 5-qubit interactions; this will be improved to 3-qubit interactions in the next section. Theorem 1.1 thus follows as a corollary from the following theorem.

THEOREM 3.1. *Given a quantum circuit on n qubits with L 2-qubit gates implementing a unitary U , and $\epsilon > 0$, there exists a 5-local adiabatic computation $AC(n + L, 2, H_{\text{init}}, H_{\text{final}}, \epsilon)$ whose running time is $\text{poly}(L, \frac{1}{\epsilon})$ and whose output (after tracing out some ancilla qubits) is ϵ -close (in trace distance) to $U|0^n\rangle$. Moreover, H_{init} and H_{final} can be computed by a polynomial time Turing machine.*

The running time we obtain here is $O(\epsilon^{-8} L^7)$.

3.1. The Hamiltonian. For our construction we use the Hamiltonian defined in [38]. Denote $|\gamma_\ell\rangle := |\alpha(\ell)\rangle \otimes |1^\ell 0^{L-\ell}\rangle^c$, where $|\alpha(\ell)\rangle$ denotes the state of the circuit after the ℓ th gate and the superscript c denotes the clock qubits. We would like to define a local Hamiltonian H_{init} with ground state $|\gamma_0\rangle = |0^n\rangle \otimes |0^L\rangle^c$, and a local Hamiltonian H_{final} with ground state $|\eta\rangle = \frac{1}{\sqrt{L+1}} \sum_{\ell=0}^L |\gamma_\ell\rangle$ as in (1). To do this, we write H_{init} and H_{final} as a sum of terms:

$$H_{\text{init}} := H_{\text{clockinit}} + H_{\text{input}} + H_{\text{clock}},$$

$$H_{\text{final}} := \frac{1}{2} \sum_{\ell=1}^L H_\ell + H_{\text{input}} + H_{\text{clock}}.$$

The terms in H_{final} (and likewise in H_{init}) are defined such that the only state whose energy (i.e., eigenvalue) is 0 is the desired ground state. This is done by assigning an energy penalty to any state that does not satisfy the required properties of the

ground state. The different terms, which correspond to different properties of the ground states, are described in the following paragraphs. The adiabatic evolution then follows the time-dependent Hamiltonian

$$(4) \quad H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}.$$

Notice that as s goes from 0 to 1, $H_{\text{clockinit}}$ is slowly replaced by $\frac{1}{2} \sum_{\ell=1}^L H_{\ell}$, while H_{input} and H_{clock} are held constant.

We now describe each of the terms. First, H_{clock} checks that the clock's state is of the form $|1^{\ell}0^{L-\ell}\rangle^c$ for some $0 \leq \ell \leq L$. This is achieved by assigning an energy penalty to any basis state on the clock qubits that contains the sequence 01,

$$H_{\text{clock}} := \sum_{\ell=1}^{L-1} |01\rangle\langle 01|_{\ell, \ell+1}^c,$$

where the subscript indicates which clock qubits the projection operates on. Note that illegal clock states are eigenstates of H_{clock} with an eigenvalue of at least 1; legal clock states have an eigenvalue of 0.

Next, H_{input} checks that if the clock is $|0^L\rangle^c$, the computation qubits must be in the state $|0^n\rangle$,

$$H_{\text{input}} := \sum_{i=1}^n |1\rangle\langle 1|_i \otimes |0\rangle\langle 0|_1^c.$$

We complete the description of H_{init} with $H_{\text{clockinit}}$, whose goal is to check that the clock's state is $|0^L\rangle^c$,

$$H_{\text{clockinit}} := |1\rangle\langle 1|_1^c.$$

CLAIM 3.2. *The state $|\gamma_0\rangle$ is a ground state of H_{init} with an eigenvalue of 0.⁷*

Proof. It is easy to verify that $H_{\text{init}}|\gamma_0\rangle = 0$. As a sum of projectors, H_{init} is positive semidefinite and hence $|\gamma_0\rangle$ is a ground state of H_{init} . \square

We now proceed to the first term in H_{final} . The Hamiltonian H_{ℓ} checks that the propagation from step $\ell-1$ to ℓ is correct, i.e., that it corresponds to the application of the gate U_{ℓ} . For $1 < \ell < L$, it is defined as

$$(5) \quad \begin{aligned} H_{\ell} := & I \otimes |100\rangle\langle 100|_{\ell-1, \ell, \ell+1}^c - U_{\ell} \otimes |110\rangle\langle 100|_{\ell-1, \ell, \ell+1}^c \\ & - U_{\ell}^{\dagger} \otimes |100\rangle\langle 110|_{\ell-1, \ell, \ell+1}^c + I \otimes |110\rangle\langle 110|_{\ell-1, \ell, \ell+1}^c. \end{aligned}$$

Intuitively, the 3-qubit terms above move the state of the clock one step forward, one step backward, or leave it unchanged. The accompanying matrices $U_{\ell}, U_{\ell}^{\dagger}$ describe the associated time evolution. For the boundary cases $\ell = 1, L$, we omit one clock qubit from these terms and define

$$(6) \quad \begin{aligned} H_1 := & I \otimes |00\rangle\langle 00|_{1,2} - U_1 \otimes |10\rangle\langle 00|_{1,2} - U_1^{\dagger} \otimes |00\rangle\langle 10|_{1,2} + I \otimes |10\rangle\langle 10|_{1,2}, \\ H_L := & I \otimes |10\rangle\langle 10|_{L-1, L} - U_L \otimes |11\rangle\langle 10|_{L-1, L} - U_L^{\dagger} \otimes |10\rangle\langle 11|_{L-1, L} + I \otimes |11\rangle\langle 11|_{L-1, L}. \end{aligned}$$

CLAIM 3.3. *The history state $|\eta\rangle$ is a ground state of H_{final} with an eigenvalue of 0.*

Proof. It is easy to verify that $H_{\text{final}}|\eta\rangle = 0$. It remains to be noticed that for all $1 \leq \ell \leq L$, H_{ℓ} is positive semidefinite and hence so is H_{final} . \square

⁷The state $|\gamma_0\rangle$ is in fact the *unique* ground state of H_{init} , as will become apparent from the proof of the global spectral gap. A similar statement holds for Claim 3.3.

3.2. Spectral Gap in a Subspace. Let \mathcal{S}_0 be the $L + 1$ -dimensional subspace spanned by $|\gamma_0\rangle, \dots, |\gamma_L\rangle$. It is easy to verify the following claim.

CLAIM 3.4. *The subspace \mathcal{S}_0 is invariant under $H(s)$, i.e., $H(s)(\mathcal{S}_0) \subseteq \mathcal{S}_0$.*

In this subsection, we show that the spectral gap of $H_{\mathcal{S}_0}(s)$, the restriction of H to \mathcal{S}_0 , is inverse polynomial in L . As mentioned in subsection 2.3, this, together with Claim 3.4, is enough to obtain a bound on the running time of the adiabatic algorithm.

Remark. Notice that both H_{clock} and H_{input} are 0 on the invariant subspace \mathcal{S}_0 . This means that Theorem 3.1 holds even if we remove the terms H_{clock} and H_{input} from both H_{init} and H_{final} . We include these terms in the Hamiltonian in order to achieve a global spectral gap (as we will see in subsection 4.1), and also for consistency with the rest of the paper.

We next lower bound the spectral gap inside \mathcal{S}_0 , using the notion of conductance. We note that, in general, applying the conductance bound requires knowing the limiting distribution of the chain, which in our case is hard since it corresponds to knowing the coefficients of the ground state for the Hamiltonians $H(s)$. We circumvent this problem by noticing that it is actually sufficient in our case to know very little about the limiting distribution of the Markov chain, namely, that it is monotone (in a certain sense to be defined).

LEMMA 3.5. *The spectral gap of the restriction of $H(s)$ to \mathcal{S}_0 satisfies $\Delta(H_{\mathcal{S}_0}(s)) = \Omega(L^{-2})$ for all $s \in [0, 1]$.*

Proof. Let us write the Hamiltonians $H_{\mathcal{S}_0, \text{init}}$ and $H_{\mathcal{S}_0, \text{final}}$ in the basis $|\gamma_0\rangle, \dots, |\gamma_L\rangle$ of \mathcal{S}_0 . Both H_{clock} and H_{input} are 0 on \mathcal{S}_0 and can thus be ignored. We have the following $(L + 1) \times (L + 1)$ matrices:

$$(7) \quad H_{\mathcal{S}_0, \text{init}} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

$$(8) \quad \begin{aligned} H_{\mathcal{S}_0, \text{final}} &= \frac{1}{2}|\gamma_0\rangle\langle\gamma_0| - \frac{1}{2}|\gamma_0\rangle\langle\gamma_1| - \frac{1}{2}|\gamma_L\rangle\langle\gamma_{L-1}| + \frac{1}{2}|\gamma_L\rangle\langle\gamma_L| \\ &\quad + \sum_{\ell=1}^{L-1} (-\frac{1}{2}|\gamma_\ell\rangle\langle\gamma_{\ell-1}| + |\gamma_\ell\rangle\langle\gamma_\ell| - \frac{1}{2}|\gamma_\ell\rangle\langle\gamma_{\ell+1}|) \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \dots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ddots & \vdots \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \vdots & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ & & & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & \dots & & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \end{aligned}$$

We now lower bound $\Delta(H_{\mathcal{S}_0}(s))$. We consider two cases.

- *The case $s < 1/3$.* Here, $H_{\mathcal{S}_0}(s)$ is sufficiently close to $H_{\mathcal{S}_0, \text{init}}$ (whose spectral gap is 1) so we can apply the following standard lemma (see, e.g., [14, p. 244]).

LEMMA 3.6 (Gerschgorin's circle theorem). *Let A be any matrix with entries a_{ij} . Consider the discs in the complex plane given by*

$$D_i = \left\{ z \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}, \quad 1 \leq i \leq n.$$

Then the eigenvalues of A are contained in $\cup D_i$ and any connected component of $\cup D_i$ contains as many eigenvalues of A as the number of discs that form this component.

For $s < 1/3$, we have that $H_{S_0}(s)_{1,1} < 1/6$ and $\sum_{j \neq 1} H_{S_0}(s)_{1,j} < 1/6$. Moreover, for any $i \neq 1$, we have that $H_{S_0}(s)_{i,i} > 5/6$ and $\sum_{j \neq i} H_{S_0}(s)_{i,j} < 1/3$. By the above lemma, we obtain that there is one eigenvalue smaller than $1/3$, while all other eigenvalues are larger than $1/2$. Hence, the spectral gap is at least $1/6$.

• *The case $s \geq 1/3$.* We note that $H_{S_0, \text{final}}$ is the Laplacian of the simple random walk [41] of a particle on a line of length $L + 1$. A standard result in Markov chain theory implies $\Delta(H_{S_0, \text{final}}) = \Omega(1/L^2)$ [41]. For $s \geq 1/3$, $H_{S_0}(s)$ is sufficiently close to $H_{S_0, \text{final}}$ to apply Markov chain techniques, as we show next.

Let $(\alpha_0, \dots, \alpha_L)^\dagger$ be the ground state of $H_{S_0}(s)$ with eigenvalue λ . Define the Hermitian matrix $G(s) = I - H_{S_0}(s)$. It is easy to see that $G(s)$ satisfies the conditions of Fact 2.3 for all $s > 0$. We obtain that the largest eigenvalue $\mu = 1 - \lambda$ of $G(s)$ is positive and nondegenerate and the corresponding eigenvector $(\alpha_0, \dots, \alpha_L)^\dagger$ has positive entries. We can now map the matrix $G(s)$ to a stochastic matrix $P(s)$ as described in subsection 2.4. The transition matrix $P(s)$ describes a random walk on the line of $L + 1$ sites (see Figure 1). Fact 2.4 implies that the limiting distribution of $P(s)$ is given by $\pi = (\alpha_0^2/Z, \dots, \alpha_L^2/Z)$, where $Z = \sum_i \alpha_i^2$.

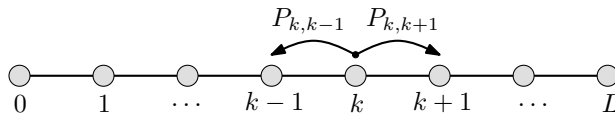


Fig. 1 The random walk of $P(s)$.

We bound the spectral gap of $P(s)$ using the conductance bound (see subsection 2.5). To do this we need to know that π is monotone. We first show the following claim.

CLAIM 3.7. *For all $0 \leq s \leq 1$, the ground state of $H_{S_0}(s)$ is monotone, namely, $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_L \geq 0$.*

Proof. The case $s = 0$ is obvious, so assume $s > 0$. We first claim that the ground state $(\alpha_0, \dots, \alpha_L)^\dagger$ of $H_{S_0}(s) = I - G(s)$ can be written as the limit

$$\frac{1}{c_0} \lim_{\ell \rightarrow \infty} (G(s)/\mu)^\ell (1, \dots, 1)^\dagger$$

for some constant $c_0 > 0$. To see this, let $|v_0\rangle, \dots, |v_L\rangle$ be an orthonormal set of eigenvectors of $G(s)$, with corresponding eigenvalues $\mu_0 \geq \mu_1 \geq \dots \geq \mu_L$. By Fact 2.3, the largest eigenvalue corresponds to a unique eigenvector, and hence we have $|v_0\rangle = (\alpha_0, \dots, \alpha_L)^\dagger$, and $\mu_0 = \mu$.

The set of eigenvectors $|v_i\rangle$ forms an orthonormal basis, and we can write $(1, \dots, 1)^\dagger$ in terms of this basis: $(1, \dots, 1)^\dagger = \sum_i c_i |v_i\rangle$. Now, we have that $(G(s)/\mu)^\ell (1, \dots, 1)^\dagger = \sum_i c_i (\frac{\mu_i}{\mu})^\ell |v_i\rangle$. By Fact 2.3 we have $|\mu_i| < \mu$ for all $i \neq 0$, and $\mu > 0$. We thus have that $\lim_{\ell \rightarrow \infty} (G(s)/\mu)^\ell (1, \dots, 1)^\dagger = c_0 |v_0\rangle$.

It is easy to check that $G(s)$ preserves monotonicity, namely, if $G(s)$ is applied to a monotone vector, the result is a monotone vector. Hence, when $G(s)/\mu$ is applied to the monotone vector $(1, \dots, 1)^\dagger$, the result is a monotone vector. Thus, $c_0|v_0\rangle$ is monotone. Finally, we observe that $c_0 > 0$. This is because c_0 is the inner product between the all 1 vector and $|v_0\rangle$, whose entries are all positive by Fact 2.3. This implies that $|v_0\rangle$ is also monotone, as desired. \square

It follows that π is also monotone. We use this and simple combinatorial arguments to prove the following claim.

CLAIM 3.8. For all $1/3 \leq s \leq 1$, $\varphi(P(s)) \geq \frac{1}{6L}$.

Proof. We show that for any nonempty $B \subseteq \{0, \dots, L\}$, $F(B)/\pi(B) \geq \frac{1}{6L}$. We consider two cases. First, assume that $0 \in B$. Let k be the smallest such that $k \in B$ but $k + 1 \notin B$. Then

$$F(B) \geq \pi_k P(s)_{k,k+1} = \pi_k \cdot \frac{\sqrt{\pi_{k+1}}}{\mu\sqrt{\pi_k}} G(s)_{k,k+1} = \frac{\sqrt{\pi_k \pi_{k+1}}}{1 - \lambda} G(s)_{k,k+1} \geq \frac{\pi_{k+1}}{1 - \lambda} G(s)_{k,k+1},$$

where the last inequality follows from the monotonicity of π . Using the definition of G and the assumption that $s \geq 1/3$ we get that $G(s)_{k,k+1} \geq 1/6$. We also have $0 < 1 - \lambda \leq 1$, where the second inequality follows from the fact that $H_{S_0}(s)$ is positive semidefinite, and the first follows from $\mu > 0$, which we previously deduced from Fact 2.3. Hence,

$$(9) \quad \frac{F(B)}{\pi(B)} \geq \frac{\pi_{k+1}}{6\pi(B)}.$$

By $\pi(B) \leq 1/2$, we have $\pi(\{k + 1, \dots, L\}) \geq 1/2$. Together with $\pi(\{k + 1, \dots, L\}) \leq L\pi_{k+1}$ we obtain $\pi_{k+1} \geq 1/(2L)$. This yields the desired bound $F(B)/\pi(B) \geq 1/(6L)$.

Now assume that $0 \notin B$ and let k be the smallest such that $k \notin B$ and $k + 1 \in B$. It is easy to see that $\pi_k P(s)_{k,k+1} = \pi_{k+1} P(s)_{k+1,k}$. Hence, using the same argument as before we can see that (9) holds in this case too. Since $B \subseteq \{k + 1, \dots, L\}$, we have $\pi(\{k + 1, \dots, L\}) \geq \pi(B)$. Hence, $\pi_{k+1} \geq \pi(B)/L$. Again, this yields the bound $F(B)/\pi(B) \geq 1/(6L)$. \square

By Theorem 2.5, we have that the spectral gap of $P(s)$ is larger than $1/(2 \cdot (6)^2 \cdot L^2)$. By subsection 2.4, we have that $\Delta(H_{S_0}) \geq \mu/(2 \cdot (6)^2 L^2)$. Finally, notice that $\mu = 1 - \lambda \geq \frac{1}{2}$, because $\lambda \leq \langle \gamma_0 | H_{S_0}(s) | \gamma_0 \rangle = \frac{s}{2} \leq \frac{1}{2}$. \square

3.3. Running Time. We now complete the proof of Theorem 3.1. Note that we have already proved something which is very close to Theorem 3.1.

CLAIM 3.9. Given a quantum circuit on n qubits with L gates, the adiabatic algorithm with H_{init} and H_{final} as defined in the previous section, with $T = O(L^6/\epsilon)$, outputs a final state that is within ℓ_2 -distance ϵ of the history state of the circuit, $|\eta\rangle$. The running time of the algorithm is $O(T \cdot L)$.

Proof. Claim 3.4 shows that S_0 is invariant under H . Hence, as mentioned in subsection 2.3, an adiabatic evolution according to H is identical to an adiabatic evolution according to H_{S_0} . Using Lemma 3.5 and Theorem 2.1 (with $\|H_{\text{init}} - H_{\text{final}}\| = O(1)$), we obtain that for T as above the final state (with global phase adjusted appropriately) is indeed ϵ -close in ℓ_2 -norm to $|\eta\rangle$. By our definition, the running time of the adiabatic algorithm is $O(T \cdot L)$ since $\|H(s)\| \leq (1 - s)\|H_{\text{init}}\| + s\|H_{\text{final}}\| = O(L + n) = O(L)$. The last equality follows from $n = O(L)$, because each qubit is assumed to participate in the computation (otherwise we can omit it). \square

In fact, one might be satisfied with this claim, which enables generating adiabatically a state which is very close to $|\eta\rangle$, instead of our desired $|\alpha(L)\rangle$. To see why

this might be sufficient to simulate quantum circuits, suppose for a moment that ϵ is 0, and the final state is exactly $|\eta\rangle$. As mentioned in the introduction, we can now measure the clock qubits of the history state, and with probability $1/L$ the outcome is $\ell = L$, which means that the state of the first register is the desired state $|\alpha(L)\rangle$. If the measurement yields another value, we repeat the adiabatic algorithm from scratch. To get $\ell = L$ with sufficiently high probability, we repeat the process $O(L)$ times, which introduces an overhead factor of L . The above discussion is also true with $\epsilon > 0$, as long as it is much smaller than $1/L$, the weight of $|\alpha(L)\rangle$ in $|\eta\rangle$.

However, strictly speaking, this is not sufficient to complete the proof of Theorem 3.1. Indeed, the theorem as stated follows our definition of the model of adiabatic computation, which allows one to perform one adiabatic evolution and then measure (and possibly trace out some qubits). Classical postprocessing such as conditioning on ℓ being equal to L , and repeating the computation if it is not, are not allowed. Hence, we need to adiabatically generate a state that is close to the final state of the circuit, $|\alpha(L)\rangle$.

This technical issue can be resolved with the following simple trick, which at the same time allows us to avoid the overhead factor of L introduced before. We simply add another $O(\frac{1}{\epsilon}L)$ identity gates to the original quantum circuit at the end of its computation. This modification ensures that the history state (after tracing out the clock qubits) is close to $|\alpha(L)\rangle$. We then apply the adiabatic simulation to this modified circuit. The following easy lemma makes this precise.

LEMMA 3.10. *Assume we can transform any given quantum circuit with L 2-qubit gates on n qubits into a k -local adiabatic computation on $n + L$ d -state particles whose output is ϵ close in ℓ_2 -norm to the history state of the quantum circuit and whose running time is $f(L, \epsilon)$ for some function f . Then we can transform any given quantum circuit with L 2-qubit gates on n qubits into a k -local adiabatic computation on $n + 2L/\epsilon$ d -state particles whose output (after tracing out some ancilla qubits) is ϵ close in trace distance to the final state of the circuit and whose running time is $f(2L/\epsilon, \epsilon/2)$.*

Proof. Given a quantum circuit on n qubits with L gates, consider the circuit obtained by appending to it $(\frac{2}{\epsilon} - 1)L$ identity gates. Let $L' = 2L/\epsilon$ be the number of gates in the modified circuit and let $|\eta\rangle$ denote its history state. By our assumption, we can transform this modified circuit into an adiabatic computation whose output is $\epsilon/2$ close in ℓ_2 -norm to $|\eta\rangle$ and whose running time is $f(L', \epsilon/2)$. Since the trace distance between two pure states is bounded from above by the ℓ_2 -distance (see, e.g., [5]), we obtain that the output of the adiabatic computation is also $\epsilon/2$ close in trace distance to $|\eta\rangle\langle\eta|$. In addition, it is easy to check that after we trace out the clock qubits from $|\eta\rangle$, we are left with a state that is $\epsilon/2$ close in trace distance to the final state of the circuit. We complete the proof by applying the triangle inequality. \square

We can now apply this lemma on the result of Claim 3.9. This completes the proof of Theorem 3.1, with the running time being $O(L^7/\epsilon^8)$.

4. Improvements and Extensions. In this section we present two extensions of the result of the previous section. We note that the techniques developed here will be used again in section 5, where we impose additional geometrical constraints on the system.

The first result is presented in subsection 4.1. It shows that the spectral gap of $H(s)$ in the *entire* Hilbert space (i.e., the global spectral gap) is nonnegligible. This shows that the adiabatic computation used in the proof of Theorem 1.1 is non-degenerate.

The second result is given in subsection 4.2. There, we show that Theorem 1.1 holds with 3-local Hamiltonians (rather than 5). The proof of this result uses techniques developed in the above spectral gap proof together with some new tools.

4.1. Global Spectral Gap.

LEMMA 4.1. *For all $0 \leq s \leq 1$, $\Delta(H(s)) = \Omega(L^{-3})$.*

Proof. Let \mathcal{S} be the subspace of dimension $(L + 1) \cdot 2^n$ spanned by all legal clock states. Observe that \mathcal{S} is preserved by $H(s)$, i.e., $H(s)(\mathcal{S}) \subseteq \mathcal{S}$. Hence, the eigenstates of $H(s)$ belong either to \mathcal{S} or to its orthogonal subspace \mathcal{S}^\perp . We can therefore analyze the spectrum of $H_{\mathcal{S}}(s)$ and of $H_{\mathcal{S}^\perp}(s)$ separately.

First, due to the term H_{clock} and the fact that all other terms are positive semidefinite, the ground energy of $H_{\mathcal{S}^\perp}(s)$ is at least 1. Second, as we will show next using Lemma 4.2, the spectral gap of $H_{\mathcal{S}}(s)$ is $\Omega(L^{-3})$. To establish the same lower bound on the spectral gap for $H(s)$, it is enough to show that the ground energy of $H_{\mathcal{S}}(s)$ is smaller than $\frac{1}{2}$. Indeed, observe that

$$\langle \gamma_0 | H_{\mathcal{S}}(s) | \gamma_0 \rangle = \langle \gamma_0 | H_{\mathcal{S}_0}(s) | \gamma_0 \rangle = s/2 \leq 1/2,$$

where the first equality holds because $|\gamma_0\rangle \in \mathcal{S}_0$ and the second follows from (7) and (8). Therefore, the smallest eigenvalue of $H_{\mathcal{S}}(s)$ is bounded from above by $1/2$. \square

LEMMA 4.2. *Let \mathcal{S} denote the subspace spanned by all legal clock states. Then the ground state of $H_{\mathcal{S}}(0)$ is $|\gamma_0\rangle$, and that of $H_{\mathcal{S}}(1)$ is $|\eta\rangle$. Moreover, for all $0 \leq s \leq 1$, $\Delta(H_{\mathcal{S}}(s)) = \Omega(L^{-3})$.*

Proof. We can write \mathcal{S} as the direct sum of 2^n orthogonal subspaces $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{2^n-1}$, defined as follows. For $0 \leq j \leq 2^n - 1$ and $0 \leq \ell \leq L$, define $|\gamma_\ell^j\rangle := |\alpha^j(\ell)\rangle \otimes |1^\ell 0^{L-\ell}\rangle$, where $|\alpha^j(\ell)\rangle$ is the state of the quantum circuit at time ℓ if the input state corresponds to the binary representation j . Note that $|\gamma_\ell^0\rangle = |\gamma_\ell\rangle$. The space \mathcal{S}_j is spanned by $\{|\gamma_0^j\rangle, \dots, |\gamma_L^j\rangle\}$. It is easy to check the following claim (see Figure 2).

CLAIM 4.3. *The Hamiltonian $H_{\mathcal{S}}(s)$ is block diagonal in the \mathcal{S}_j 's.*

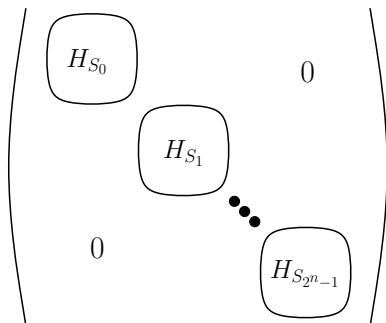


Fig. 2 $H_{\mathcal{S}}(s)$ is block diagonal.

By Claims 3.2, 3.3, and 4.3 and Lemma 3.5, it suffices to argue that the ground energy of $H_{\mathcal{S}_j}(s)$ for any $j \neq 0$ is larger than the ground energy of $H_{\mathcal{S}_0}(s)$ by at least $\Omega(1/L^3)$. Essentially, this follows from the penalty given by the term H_{input} to nonzero input states. The proof, however, is slightly subtle since H_{input} assigns a penalty only to states $|\gamma_\ell^j\rangle$ with $\ell = 0$.

Notice that

$$H_{S_j}(s) = H_{S_0}(s) + H_{S_j, \text{input}}.$$

Moreover, for $1 \leq j \leq 2^n - 1$, $H_{S_j, \text{input}}$ is diagonal, with its top-left element at least 1 (it actually equals the number of 1's in the binary representation of j) and all other diagonal elements zero. Hence, if we define M as

$$M := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

then $H_{S_j, \text{input}} - M$ is positive definite and therefore we can lower bound the ground energy of $H_{S_j}(s)$ with the ground energy of $H_{S_0}(s) + M$. For this, we apply the following geometrical lemma by Kitaev (Lemma 14.4 in [38]).

LEMMA 4.4. *Let H_1, H_2 be two Hamiltonians with ground energies a_1, a_2 , respectively. Suppose that for both Hamiltonians the difference between the energy of the (possibly degenerate) ground space and the next highest eigenvalue is larger than Λ , and that the angle between the two ground spaces is θ . Then the ground energy of $H_1 + H_2$ is at least $a_1 + a_2 + 2\Lambda \sin^2(\theta/2)$.*

We now apply this lemma to $H_{S_0}(s)$ and M . By Lemma 3.5, the spectral gap of $H_{S_0}(s)$ is $\Omega(1/L^2)$. The spectral gap of M is clearly 1. Moreover, using Claim 3.7, we obtain that the angle between the two ground spaces satisfies $\cos(\theta) \leq 1 - 1/L$ by the monotonicity property of the ground state of $H_{S_0}(s)$ (see Claim 3.7). It follows that the ground energy of $H_{S_j}(s)$ is higher by at least $\Omega(1/L^3)$ than that of $H_{S_0}(s)$. \square

Remark. Notice that we only used the following properties of H_{input} : its restriction to S_0 is 0 and its restriction to S_j for any $j \neq 0$ is a diagonal matrix in the basis $|\gamma_0^j\rangle, \dots, |\gamma_L^j\rangle$ whose top-left entry is at least 1 and all other entries are nonnegative. This observation will be useful in section 5.

4.2. Three-Local Hamiltonian. We now show that adiabatic computation with 3-local Hamiltonians is sufficient to simulate standard quantum computations.

THEOREM 4.5. *Given a quantum circuit on n qubits with L 2-qubit gates implementing a unitary U , and $\epsilon > 0$, there exists a 3-local adiabatic computation $AC(n + L, 2, H_{\text{init}}, H_{\text{final}}, \epsilon)$ whose running time is $\text{poly}(L, \frac{1}{\epsilon})$ and whose output state is ϵ -close (in trace distance) to $U|0^n\rangle$. Moreover, H_{init} and H_{final} can be computed by a polynomial time Turing machine.*

The proof of this theorem builds on techniques developed in previous subsections.

4.2.1. The Hamiltonian. Consider the Hamiltonian constructed in subsection 3.1. Notice that all terms except H_ℓ are already 3-local (some are even 2-local or 1-local). In order to obtain a 3-local Hamiltonian, we remove two clock qubits from the 5-local terms in H_ℓ and leave only the ℓ th clock qubit. More precisely, for $1 < \ell < L$ define

$$H'_\ell := I \otimes |100\rangle\langle 100|_{\ell-1, \ell, \ell+1}^c - U_\ell \otimes |1\rangle\langle 0|_\ell^c - U_\ell^\dagger \otimes |0\rangle\langle 1|_\ell^c + I \otimes |110\rangle\langle 110|_{\ell-1, \ell, \ell+1}^c.$$

For the boundary cases $l = 1, L$ we define

$$\begin{aligned} H'_1 &:= I \otimes |00\rangle\langle 00|_{1,2}^c - U_1 \otimes |1\rangle\langle 0|_1^c - U_1^\dagger \otimes |0\rangle\langle 1|_1^c + I \otimes |10\rangle\langle 10|_{1,2}^c, \\ H'_L &:= I \otimes |10\rangle\langle 10|_{L-1, L}^c - U_L \otimes |1\rangle\langle 0|_L^c - U_L^\dagger \otimes |0\rangle\langle 1|_L^c + I \otimes |11\rangle\langle 11|_{L-1, L}^c. \end{aligned}$$

Note that because of the terms $|1\rangle\langle 0|^c$ and $|0\rangle\langle 1|^c$, these Hamiltonians no longer leave the subspace \mathcal{S} invariant. To mend this, we assign a much larger energy penalty to illegal clock states. As we will see soon, this makes the lower part of the spectrum of our Hamiltonians behave essentially like in their restriction to \mathcal{S} . Set $J = \epsilon^{-2}L^6$ and define

$$H'_{\text{init}} := H_{\text{clockinit}} + H_{\text{input}} + J \cdot H_{\text{clock}},$$

$$H'_{\text{final}} := \frac{1}{2} \sum_{\ell=1}^L H'_\ell + H_{\text{input}} + J \cdot H_{\text{clock}}.$$

The Hamiltonian we use here is thus

$$H'(s) = (1 - s)H'_{\text{init}} + sH'_{\text{final}}.$$

Essentially the same proof as that of Claim 3.2 shows that $|\gamma_0\rangle$ is a ground state of H'_{init} . However, it turns out that $|\eta\rangle$ is no longer a ground state of H'_{final} (the proof of Claim 3.3 does not apply since H'_ℓ is no longer positive semidefinite). However, as we shall see later, $|\eta\rangle$ is very close to the ground state of H'_{final} .

4.2.2. The Spectral Gap. Our first claim is that, when restricted to \mathcal{S} , H' and H are identical.

CLAIM 4.6. For any $0 \leq s \leq 1$, $H_S(s) = H'_S(s)$.

Proof. Let Π_S be the orthogonal projection on \mathcal{S} . Then our goal is to show that $\Pi_S H(s) \Pi_S = \Pi_S H'(s) \Pi_S$. The only difference between $H(s)$ and $H'(s)$ is the factor of J in H_{clock} , and that the H_ℓ terms are replaced by H'_ℓ . We note that $H_{\mathcal{S},\text{clock}}$ is zero. Hence, it suffices to show that for all $1 \leq \ell \leq L$,

$$\Pi_S H_\ell \Pi_S = \Pi_S H'_\ell \Pi_S.$$

For this, observe that for any $1 < \ell < L$,

$$\Pi_S |1\rangle\langle 0|_\ell^c \Pi_S = |1^\ell 0^{L-\ell}\rangle\langle 1^{\ell-1} 0^{L-(\ell-1)}|^c = \Pi_S |110\rangle\langle 100|_{\ell-1,\ell,\ell+1}^c \Pi_S$$

and similarly for $|0\rangle\langle 1|_\ell^c$. A similar statement holds for $\ell = 1, L$ with the right-hand term modified appropriately. \square

Lemma 4.2 and Claim 4.6 imply that $\Delta(H'_S(s)) = \Omega(L^{-3})$. We now want to deduce from this a lower bound on $\Delta(H'(s))$, without the restriction to \mathcal{S} . For this we use the following claim. Essentially, it says that if J is large enough, then the lower part of the spectrum of $H'(s)$ is similar to that of $H'_S(s)$. More precisely, it shows that the lowest eigenvalues, the second lowest eigenvalues, and the ground states of the two Hamiltonians are close. Intuitively, this holds since the energy penalty given to states in \mathcal{S}^\perp , the orthogonal space to \mathcal{S} , is very high and hence any eigenvector with low eigenvalue must be almost orthogonal to \mathcal{S}^\perp (and hence almost inside \mathcal{S}). We note that a similar lemma was used in [36] in the context of QMA-complete problems.

LEMMA 4.7. Let $H = H_1 + H_2$ be the sum of two Hamiltonians operating on some Hilbert space $\mathcal{H} = \mathcal{S} + \mathcal{S}^\perp$. The Hamiltonian H_2 is such that \mathcal{S} is a zero eigenspace and the eigenvectors in \mathcal{S}^\perp have an eigenvalue of at least $J > 2K$, where $K = \|H_1\|$. Let a and b be the lowest and second lowest eigenvalues of H_S , and let a' and b' be the corresponding quantities for H . Then the lowest eigenvalue of H satisfies $a - \frac{K^2}{J-2K} \leq a' \leq a$ and the second lowest eigenvalue of H satisfies $b' \geq b - \frac{K^2}{J-2K}$. If, moreover, $b > a$, then the ground states $|\xi\rangle, |\xi'\rangle$ of H_S, H , respectively, satisfy

$$|\langle \xi | \xi' \rangle|^2 \geq 1 - \frac{K^2}{(b-a)(J-2K)}.$$

Proof. First, we show that $a' \leq a$. Using $H_2|\xi\rangle = 0$,

$$\langle \xi | H | \xi \rangle = \langle \xi | H_1 | \xi \rangle + \langle \xi | H_2 | \xi \rangle = a,$$

and hence H must have an eigenvector of eigenvalue at most a .

We now show the lower bound on a' . We can write any unit vector $|v\rangle \in \mathcal{H}$ as $|v\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle$, where $|v_1\rangle \in \mathcal{S}$ and $|v_2\rangle \in \mathcal{S}^\perp$ are two unit vectors and α_1, α_2 are two nonnegative reals satisfying $\alpha_1^2 + \alpha_2^2 = 1$. Then we have

$$\begin{aligned} \langle v | H | v \rangle &\geq \langle v | H_1 | v \rangle + J\alpha_2^2 \\ &= (1 - \alpha_2^2)\langle v_1 | H_1 | v_1 \rangle + 2\alpha_1\alpha_2\operatorname{Re}\langle v_1 | H_1 | v_2 \rangle + \alpha_2^2\langle v_2 | H_1 | v_2 \rangle + J\alpha_2^2 \\ &\geq \langle v_1 | H_1 | v_1 \rangle - K\alpha_2^2 - 2K\alpha_2 - K\alpha_2^2 + J\alpha_2^2 \\ &= \langle v_1 | H_1 | v_1 \rangle + (J - 2K)\alpha_2^2 - 2K\alpha_2, \end{aligned}$$

where we used $\alpha_1^2 = 1 - \alpha_2^2$ and $\alpha_1 \leq 1$. Since $(J - 2K)\alpha_2^2 - 2K\alpha_2$ is minimized for $\alpha_2 = K/(J - 2K)$, we have

$$(10) \quad \langle v | H | v \rangle \geq \langle v_1 | H_1 | v_1 \rangle - \frac{K^2}{J - 2K}.$$

We obtain the required lower bound by noting that $\langle v_1 | H_1 | v_1 \rangle \geq a$.

Consider now the two-dimensional space \mathcal{L} spanned by the two eigenvectors of H corresponding to a' and b' . For any unit vector $|v\rangle \in \mathcal{L}$ we have $\langle v | H | v \rangle \leq b'$. Hence, if \mathcal{L} contains a vector $|v\rangle$ orthogonal to \mathcal{S} , then we have $b' \geq \langle v | H | v \rangle \geq J - K > K \geq b$ and we are done. Otherwise, the projection of \mathcal{L} on \mathcal{S} must be a two-dimensional space. Being two-dimensional, this space must contain a vector orthogonal to $|\xi\rangle$. Let $|v\rangle$ be a vector in \mathcal{L} whose projection on \mathcal{S} is orthogonal to $|\xi\rangle$. By (10), $b' \geq \langle v | H | v \rangle \geq b - \frac{K^2}{J - 2K}$, as required.

Finally, let $\beta = |\langle \xi | \xi' \rangle|^2$. Then we can write $|\xi\rangle = \sqrt{\beta}|\xi'\rangle + \sqrt{1 - \beta}|\xi'^\perp\rangle$ for some unit vector $|\xi'^\perp\rangle$ orthogonal to $|\xi'\rangle$. Since $|\xi'\rangle$ is an eigenvector of H ,

$$\begin{aligned} a &= \langle \xi | H | \xi \rangle = \beta\langle \xi' | H | \xi' \rangle + (1 - \beta)\langle \xi'^\perp | H | \xi'^\perp \rangle \\ &\geq \beta a' + (1 - \beta)b' \\ &\geq \beta\left(a - \frac{K^2}{J - 2K}\right) + (1 - \beta)\left(b - \frac{K^2}{J - 2K}\right) \\ &= a + (1 - \beta)(b - a) - \frac{K^2}{J - 2K}. \end{aligned}$$

Rearranging, we obtain the required bound. \square

We can now bound the spectral gap of $H'(s)$.

LEMMA 4.8. *For all $0 \leq s \leq 1$, $\Delta(H'(s)) = \Omega(L^{-3})$.*

Proof. We apply Lemma 4.7 by setting $H_2 = J \cdot H_{clock}$ and H_1 to be the remaining terms such that $H'(s) = H_1 + H_2$. Note that Lemma 4.7 implies that the spectral gap of $H'(s)$ is smaller than that of $H'_S(s)$ (which is $\Omega(1/L^3)$ by Lemma 4.2) by at most $K^2/(J - 2K)$. But it is easy to see that $K = O(L)$, due to the fact that H_1 consists of $O(L)$ terms, each of constant norm. The result follows since $J = \epsilon^{-2}L^6$. \square

This shows the desired bound on the spectral gap. Before we complete the proof, we must show that the final ground state is close to the history state.

LEMMA 4.9. *The ground state of $H'(1)$ is ϵ -close to $|\eta\rangle$.*

Proof. Apply Lemma 4.7 as in the proof of Lemma 4.8, for the case $s = 1$. We obtain that the inner product squared between the ground state of $H'(1)$ and $|\eta\rangle$ is at least $1 - \delta$, with $\delta = \frac{K^2}{(b-a)(J-2K)} = O(L^{-1}\epsilon^2)$, where we have used $K = O(L)$, $J = \epsilon^{-2}L^6$, and $b - a = \Omega(1/L^3)$ by Lemma 4.2. This implies that the ℓ_2 -distance between the ground state of $H'(1)$ and $|\eta\rangle$ is $O(\epsilon/\sqrt{L}) \leq \epsilon$. \square

We now complete the proof of Theorem 4.5. The adiabatic algorithm starts with $|\gamma_0\rangle$ and evolves according to $H'(s)$ for $T = \Theta(L^{10}/\epsilon)$. Such a T satisfies the adiabatic condition (see (2)), using $\|H'_{\text{final}} - H'_{\text{init}}\| = O(L)$. By Theorem 2.1 the final state is ϵ -close in ℓ_2 -distance to the ground state of H'_{final} . Lemma 4.9 implies that this state is ϵ -close in ℓ_2 -distance to $|\eta\rangle$. Using the triangle inequality we note that the output of the adiabatic computation is 2ϵ -close to $|\eta\rangle$. The running time of this algorithm is $O(T \cdot J \cdot L) = O(T \cdot \epsilon^{-2}L^7) = O(L^{17}/\epsilon^3)$.

We can now apply Lemma 3.10 to obtain a modified adiabatic computation whose output state after tracing out the clock qubits is ϵ -close in trace distance to $U|0^n\rangle$. The running time is $O(L^{17}/\epsilon^{20})$.

5. Two-Local Hamiltonians on a Two-Dimensional Lattice. In this section we prove Theorem 1.3. We simulate a given quantum circuit by an adiabatic evolution of a system of 6-state quantum particles arranged on a two-dimensional grid. More precisely, we prove the following theorem.

THEOREM 5.1. *Given a quantum circuit on n qubits with L 2-qubit gates implementing a unitary U , and $\epsilon > 0$, there exists a 2-local adiabatic computation $AC(\text{poly}(n, L), 6, H_{\text{init}}, H_{\text{final}}, \epsilon)$ such that H_{init} and H_{final} involve only nearest neighbors on a two-dimensional grid. Moreover, the running time of this algorithm is $\text{poly}(L, \frac{1}{\epsilon})$, and its output (after performing a partial measurement on each particle) is ϵ -close (in trace distance) to $U|0^n\rangle$. Finally, H_{init} and H_{final} can be computed by a polynomial time Turing machine.*

As mentioned in the introduction, the main problem in moving to a two-dimensional grid is the notion of a clock. In the constructions of the previous section, the clock is represented by an additional register that counts the clock steps in unary representation. The terms H_ℓ , which check the correct propagation in the ℓ th time step, interact between the ℓ th qubit of the clock and the corresponding qubits on which U_ℓ operates. If we want to restrict the interaction to nearest neighbors in two dimensions using this idea, then no matter how the clock qubits are arranged on the grid, we run into problems interacting the qubits with the corresponding clock qubits in a local way. The solution to this problem lies in the way we represent the clock. Instead of using an extra register, we embed the clock into the same particles that perform the computation by defining the notion of a *shape* of a state, to be defined later. We then create a sequence of legal shapes and show how states can evolve from one legal shape to another.

Although the construction of this section is more involved than the ones of the previous section, its analysis follows almost immediately from the analysis carried out in Theorem 4.5. To achieve this, we make sure that the Hamiltonians and some relevant subspaces are as similar as possible to those in the previous section.

5.1. Assumptions on the Input Circuit. To simplify the construction of our adiabatic evolution, we first assume without loss of generality that the quantum circuit we wish to simulate has a particular layout of its gates. Namely, it consists of R rounds, where each round is composed of n nearest neighbor gates (some can be the identity gate), followed by n identity gates, as in Figure 3. More specifically, the first

gate in each round is a 1-qubit gate applied to the first qubit. For $i = 2, \dots, n$, the i th gate is a 2-qubit gate applied to qubits $i - 1$ and i . For $i = n + 1, \dots, 2n$ the i th gate is an identity gate applied to the $(2n + 1 - i)$ th qubit. These identity gates are included for convenience of notation. Any circuit can be transformed to such a form by introducing extra identity and swap gates. Let $L = 2nR$ be the total number of gates in the circuit so obtained. Clearly, L is at most polynomially larger than the number of gates in the original circuit.

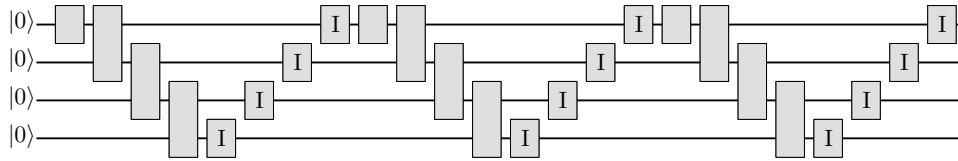


Fig. 3 The modified circuit with $R = 3$.

5.2. The Particles of the Adiabatic Quantum System. The adiabatic computation is performed on 6-state particles, arranged on a two-dimensional square lattice with n rows and $R + 1$ columns. We number the rows from 1 (top) to n (bottom) and the columns from 0 (left) to R (right). Columns number 0 and 1 are used to simulate the first round of the circuit. Column numbers 1 and 2 are used for the second round of computation, and so on. We denote the six internal states of a particle by $|\circ\rangle, |\oplus\rangle, |\ominus\rangle, |\otimes\rangle, |\oplus\rangle, |\ominus\rangle$, and $|\otimes\rangle$. These six states are divided into four *phases*: the *unborn phase* $|\circ\rangle$, the *first phase* $|\oplus\rangle, |\ominus\rangle$, the *second phase* $|\oplus\rangle, |\ominus\rangle$, and the *dead phase* $|\otimes\rangle$. The two states in the first phase and the two states in the second phase correspond to computational degrees of freedom, namely, to the $|0\rangle$ and $|1\rangle$ states of a qubit. We write $|\oplus\rangle$ to denote an arbitrary state in the subspace spanned by $|\oplus\rangle$ and $|\ominus\rangle$. Similarly, $|\ominus\rangle$ denotes a state in the space spanned by $|\oplus\rangle$ and $|\ominus\rangle$. The phases are used to define the *shape* of the basis states. A shape of a basis state is simply an assignment of one of the four phases to each particle, ignoring the computational degrees of freedom inside the first and second phases. These shapes will be used instead of the clock states of the previous section.

5.3. Geometrical Clock. We now describe the way we represent a clock using shapes. In the previous constructions, the space \mathcal{S} of dimension $2^n(L + 1)$ was the ground space of the clock, i.e., the space spanned by legal clock states. Inside the clock register there were $L + 1$ legal clock states. Note that each such clock state can be described, essentially, in a geometric way by the “shape” of the clock particles: how many 1’s precede how many 0’s.

We now describe the corresponding subspaces involved in our construction for the two-dimensional case. For each $0 \leq \ell \leq L$, we have a 2^n -dimensional subspace corresponding to that clock state. Each of these $L + 1$ subspaces can be described by its *shape*, that is, a setting of one of the four phases to each particle. See Figure 4 for an illustration with $n = 6, R = 6$. The six shapes shown correspond to clock states $\ell = 0, \ell = 4n, \ell = 4n + 3, \ell = 5n + 2, \ell = 6n$, and $\ell = 2nR$, respectively. Notice that each shape has exactly n particles in the first or second phase. Hence, the dimension of the subspace induced by each shape is 2^n . As ℓ goes from 0 to L , the shape changes from that shown in Figure 4a to that shown in Figure 4f. The locations at which the changes occur form a snake-like pattern winding down and up the lattice, following the layout of the gates in the input circuit (see Figure 3 for an example with $R = 3$).

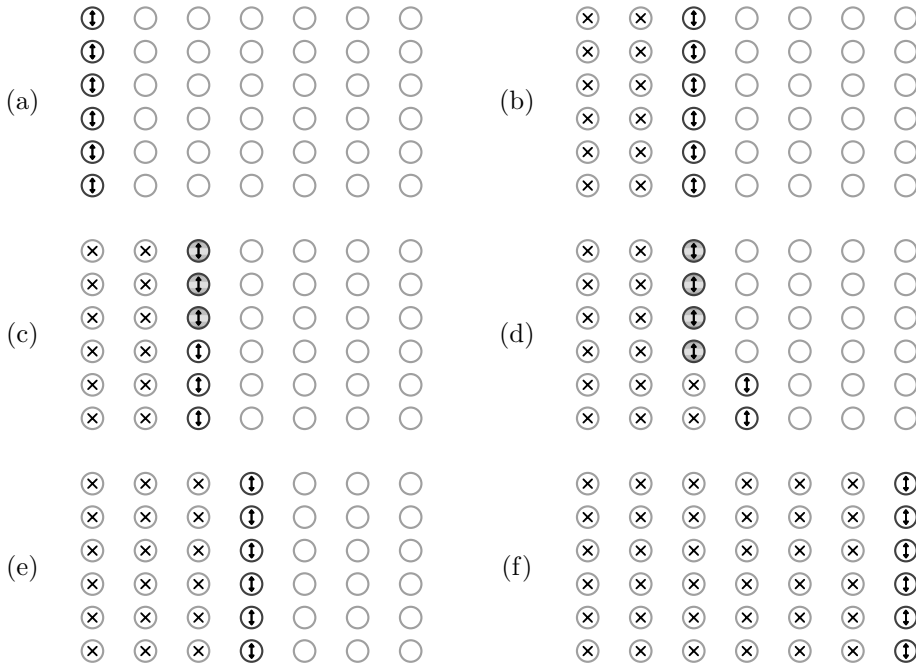


Fig. 4 Legal shapes of clock states for $n = 6$, $R = 6$ at different stages of the computation (see subsection 5.3).

We now describe the legal shapes more formally.

1. The shape corresponding to clock state $\ell = 2nr + k$ for $0 \leq k \leq n$ has its r leftmost columns in the dead phase. The top k particles in the $r + 1$ st column are in their second phase while the bottom $n - k$ are in the first phase. Particles in the remaining $R - r$ columns are all in the unborn phase.
2. The shape corresponding to clock state $\ell = 2nr + n + k$ for $1 \leq k \leq n - 1$ has, as before, its r leftmost columns in the dead phase. The $r + 1$ st column has its $n - k$ topmost particles in the second phase, and its remaining k particles in the dead phase. The $r + 2$ nd column has its $n - k$ topmost particles in the unborn phase and its remaining k particles in the first phase. All remaining particles are in the unborn phase.

The subspace \mathcal{S} is defined as the $(L + 1)2^n$ -dimensional space spanned by all legal shapes. As in previous sections we partition \mathcal{S} into 2^n subspaces \mathcal{S}_j . Each subspace \mathcal{S}_j is spanned by $L + 1$ orthogonal states $|\gamma_0^j\rangle, \dots, |\gamma_L^j\rangle$, defined as follows. For each $0 \leq \ell \leq L$ and $0 \leq j \leq 2^n - 1$, the shape of $|\gamma_\ell^j\rangle$ corresponds to ℓ . The state of the n active particles (i.e., those in either the first or second phase), when read from top to bottom, corresponds to the state of the circuit after the first ℓ gates are applied to an initial state corresponding to the binary representation of j ; i.e., it corresponds to the state $U_\ell \cdot U_{\ell-1} \cdots U_1 |j\rangle$. More precisely, these particles are in a superposition obtained by mapping this state to the state of the n active particles in the following way: $|0\rangle$ to \uparrow (or \downarrow for a second phase particle) and $|1\rangle$ to \downarrow (or \uparrow for a second phase particle). We often denote $|\gamma_\ell^0\rangle$, which corresponds to the all 0 input, by $|\gamma_\ell\rangle$. For example, $|\gamma_0\rangle$ is shown in Figure 5.

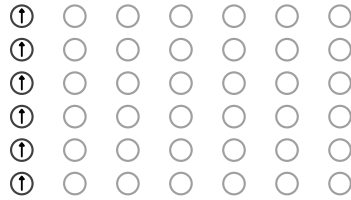


Fig. 5 *The initial state.*

At the risk of being somewhat redundant, let us now give an alternative description of the states $|\gamma_0^j\rangle, \dots, |\gamma_L^j\rangle$. This description is more helpful in understanding the Hamiltonians H_ℓ'' which we will define shortly. Consider a state $|\gamma_\ell^j\rangle$ for some $\ell = 2rn$. The n particles in the r th column are in their first phase and their computational degrees of freedom correspond to the state of the circuit's qubits at the beginning of the r th round. Particles to the left of this column are dead, those to the right of this column are unborn. The state $|\gamma_{\ell+1}^j\rangle$ is obtained from $|\gamma_\ell^j\rangle$ by changing the topmost particle in the r th column to a second phase particle and applying the first gate in the r th round (a 1-qubit gate) to its computational degrees of freedom. Next, the state $|\gamma_{\ell+2}^j\rangle$ is obtained from $|\gamma_{\ell+1}^j\rangle$ by changing the second particle from above in the r th column to a second phase particle and applying the second gate in the r th round (a 2-qubit gate) to both this particle and the one on top of it. We continue in a similar fashion until we reach $|\gamma_{\ell+n}^j\rangle$, in which the entire r th column is in the second phase. We refer to these steps as the downward stage.

Next, let us describe the upward stage. The state $|\gamma_{\ell+n+1}^j\rangle$ is obtained from $|\gamma_{\ell+n}^j\rangle$ by “moving” the bottommost particle in the r th column one location to the right. More precisely, the bottommost particle changes to the dead phase and the one to the right of it changes to the first phase. The computational degrees of freedom are the same in both states. This corresponds to the fact that the $n + 1$ st gate in a round of the circuit is the identity gate.⁸ Continuing in a similar fashion, we see that the upwards stage ends in the state $|\gamma_{\ell+n+n}^j\rangle = |\gamma_{2(r+1)n}^j\rangle$, which matches the above description of the first state in a round.

5.4. The Hamiltonian. We now construct a 2-local Hamiltonian that guarantees correct propagation from one γ state to the next. In other words, the Hamiltonian has the history state, namely, the superposition over all the γ states, as its ground state. The construction of this Hamiltonian should be more or less clear by now, where the only subtleties are due to edge cases.

The initial and final Hamiltonians are defined as

$$H_{\text{init}}'' := H_{\text{clockinit}}'' + H_{\text{input}}'' + J \cdot H_{\text{clock}}'',$$

$$H_{\text{final}}'' := \frac{1}{2} \sum_{\ell=1}^L H_\ell'' + H_{\text{input}}'' + J \cdot H_{\text{clock}}'',$$

where $J = \epsilon^{-2} \cdot L^6$. These Hamiltonians are chosen to be as similar as possible to the corresponding Hamiltonians in previous sections. For example, H_{clock}'' has as its

⁸We could allow arbitrary 1-qubit gates here instead of identity gates. This leads to a slightly more efficient construction but also to more cumbersome Hamiltonians.

ground space the space of legal clock states, \mathcal{S} . As before, it allows us to essentially project all other Hamiltonians on \mathcal{S} by assigning a large energy penalty to states with illegal shape. Also, the Hamiltonians H''_ℓ (once projected to \mathcal{S}) check correct propagation from one step to the next. Other terms also serve similar roles as before.

Let us start with the simplest terms. Define

$$H''_{\text{input}} := \sum_{i=1}^n (|\uparrow\rangle\langle\uparrow|)_{i,1}.$$

The indices indicate the row and column of the particle on which the Hamiltonian operates. This Hamiltonian checks that none of the particles in the leftmost column are in $|\uparrow\rangle$. Then, define

$$H''_{\text{clockinit}} = (I - |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_{1,1}.$$

This Hamiltonian checks that the top-left particle is in a $|\uparrow\rangle$ state. The remaining terms are described in the following subsections.

5.4.1. The Clock Hamiltonian. The shapes we define satisfy the following important property: there exist 2-local conditions that guarantee that a shape is legal. This allows us to define a 2-local clock Hamiltonian, H''_{clock} , whose ground space is exactly \mathcal{S} , the $(L + 1)2^n$ -dimensional space spanned by all legal shapes.

CLAIM 5.2. *A shape is legal if and only if it contains none of the forbidden configurations of Table 1.*

Proof. It is easy to check that any legal shape contains none of the forbidden configurations. For the other direction, consider any shape that contains none of these configurations. Observe that each row must be of the form $\otimes^* [\uparrow, \downarrow] \circ^*$; that is, it starts with a sequence of zero or more \otimes , it then contains either \uparrow or \downarrow , and then ends with a sequence of zero or more \circ . Columns can be of three different forms. Read from top to bottom, it is either $\uparrow^* \downarrow^*$, $\downarrow^* \otimes^*$, or $\circ^* \uparrow^*$. It is now easy to verify that such a shape must be one of the legal shapes. \square

Using this claim, we can define a 2-local nearest-neighbor Hamiltonian that guarantees a legal shape. For example, if the rule forbids a particle at location (i, j) in state \circ to the left of a particle at location $(i, j + 1)$ in state \otimes , then the corresponding

Table 1 Local rules for basis state to be in \mathcal{S} .

Forbidden	Guarantees that
$\circ\uparrow, \circ\downarrow, \circ\otimes$	\circ is to the right of all other qubits
$\circ\otimes, \uparrow\otimes, \downarrow\otimes$	\otimes is to the left of all other qubits
$\circ\otimes, \otimes\circ$	\circ and \otimes are not horizontally adjacent
$\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$	only one of \uparrow, \downarrow per row
$\circ \quad \uparrow \quad \otimes$ $\uparrow, \downarrow, \downarrow$	only \uparrow above \downarrow
$\uparrow \quad \downarrow \quad \uparrow$ $\circ, \downarrow, \otimes$	only \downarrow below \uparrow
$\circ \quad \otimes$ \otimes, \circ	\circ and \otimes are not vertically adjacent
$\uparrow \quad \otimes$ \circ, \downarrow	no \circ below \downarrow and no \uparrow below \otimes

term in the Hamiltonian is $(|\circ, \otimes\rangle\langle\circ, \otimes|)_{(i,j),(i,j+1)}$. Summing over all the forbidden configurations of Table 1 and over all relevant pairs of particles, we have

$$H''_{\text{clock}} := \sum_{r \in \text{rules}} H_r.$$

Note that the ground space of H''_{clock} is the $(L+1)2^n$ -dimensional space \mathcal{S} .

5.4.2. The Propagation Hamiltonian. The choice of legal shapes has the following important property: the shape of ℓ and that of $\ell+1$ differ in at most two locations. This means that for any ℓ and j , the shape of $|\gamma_{\ell-1}^j\rangle$ and that of $|\gamma_{\ell}^j\rangle$ differ in at most two locations. Moreover, if we consider the state of the n active particles in both states, we see that these differ on at most two particles, namely, those on which the ℓ th gate in the circuit acts. Crucially—and this is where we use our assumption on the form of the circuit (Figure 3)—the particle(s) on which the ℓ th gate acts are at the same location as the particle(s) whose phase changes. It is this structure that allows us to define the Hamiltonians H''_{ℓ} . These Hamiltonians act on two particles and “simultaneously” advance the clock (by changing the shape) and advance the computational state (by modifying the state of the active particles). Since $|\gamma_{\ell}\rangle$ differs from $|\gamma_{\ell-1}\rangle$ in at most two adjacent lattice sites, this can be done using a two-body nearest neighbor Hamiltonian.

The definition of H''_{ℓ} depends on whether ℓ is in the downward phase (i.e., is of the form $2rn+k$ for $1 \leq k \leq n$) or in the upward phase (i.e., is of the form $2rn+n+k$ for $1 \leq k \leq n$). We first define H''_{ℓ} for the upward phase. Assume $\ell = 2rn+n+k$ for some $0 \leq r < R, 1 < k < n$ and let $i = n-k+1$ be the row in which $|\gamma_{\ell-1}\rangle$ and $|\gamma_{\ell}\rangle$ differ. Then

$$\begin{aligned} H''_{\ell} := & \left| \begin{array}{c} \uparrow \\ \otimes \end{array} \right\rangle \left\langle \begin{array}{c} \uparrow \\ \otimes \end{array} \right|_{i+1,r}^{i,r} + \left| \begin{array}{c} \circ \\ \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \uparrow \end{array} \right|_{i,r+1}^{i-1,r+1} - (|\uparrow, \circ\rangle\langle\otimes, \uparrow| + |\otimes, \uparrow\rangle\langle\uparrow, \circ|)_{(i,r)(i,r+1)} \\ & + \left| \begin{array}{c} \downarrow \\ \otimes \end{array} \right\rangle \left\langle \begin{array}{c} \downarrow \\ \otimes \end{array} \right|_{i+1,r}^{i,r} + \left| \begin{array}{c} \circ \\ \downarrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \downarrow \end{array} \right|_{i,r+1}^{i-1,r+1} - (|\downarrow, \circ\rangle\langle\otimes, \downarrow| + |\otimes, \downarrow\rangle\langle\downarrow, \circ|)_{(i,r)(i,r+1)}. \end{aligned}$$

The first line corresponds to changing the state $|\uparrow, \circ\rangle$ into $|\otimes, \uparrow\rangle$. The second line is similar for $|\downarrow, \circ\rangle$ and $|\otimes, \downarrow\rangle$. The purpose of the first two terms in each line is the same as that of $|100\rangle\langle 100|^c$ and $|110\rangle\langle 110|^c$ in H_{ℓ} from previous sections.⁹ The difference is that here, to uniquely identify the current clock state, we need to consider particles on top of each other. The remaining terms in each line correspond to $|100\rangle\langle 110|^c$ and $|100\rangle\langle 110|^c$ in H_{ℓ} .

For the case $k=1, n$, the definition is

$$\begin{aligned} H''_{2rn+n+1} := & \left| \begin{array}{c} \uparrow \\ \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \uparrow \\ \uparrow \end{array} \right|_{n,r}^{n-1,r+1} + \left| \begin{array}{c} \circ \\ \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \uparrow \end{array} \right|_{n,r+1}^{n-1,r+1} - (|\uparrow, \circ\rangle\langle\otimes, \uparrow| + |\otimes, \uparrow\rangle\langle\uparrow, \circ|)_{(n,r)(n,r+1)} \\ & + \left| \begin{array}{c} \downarrow \\ \downarrow \end{array} \right\rangle \left\langle \begin{array}{c} \downarrow \\ \downarrow \end{array} \right|_{n,r}^{n-1,r+1} + \left| \begin{array}{c} \circ \\ \downarrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \downarrow \end{array} \right|_{n,r+1}^{n-1,r+1} - (|\downarrow, \circ\rangle\langle\otimes, \downarrow| + |\otimes, \downarrow\rangle\langle\downarrow, \circ|)_{(n,r)(n,r+1)}, \\ H''_{2rn+2n} := & \left| \begin{array}{c} \uparrow \\ \otimes \end{array} \right\rangle \left\langle \begin{array}{c} \uparrow \\ \otimes \end{array} \right|_{2,r}^{1,r} + \left| \begin{array}{c} \circ \\ \uparrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \uparrow \end{array} \right|_{1,r+1} - (|\uparrow, \circ\rangle\langle\otimes, \uparrow| + |\otimes, \uparrow\rangle\langle\uparrow, \circ|)_{(1,r)(1,r+1)} \\ & + \left| \begin{array}{c} \downarrow \\ \otimes \end{array} \right\rangle \left\langle \begin{array}{c} \downarrow \\ \otimes \end{array} \right|_{2,r}^{1,r} + \left| \begin{array}{c} \circ \\ \downarrow \end{array} \right\rangle \left\langle \begin{array}{c} \circ \\ \downarrow \end{array} \right|_{1,r+1} - (|\downarrow, \circ\rangle\langle\otimes, \downarrow| + |\otimes, \downarrow\rangle\langle\downarrow, \circ|)_{(1,r)(1,r+1)}. \end{aligned}$$

⁹There are other (equally good) ways to define these terms. For example, it is possible to define them so that they both act on the r th column.

For the downward stage, H''_ℓ checks that a gate is applied correctly. For $\ell = 2nr+k$ and $1 < k < n$ we define

$$H''_\ell := \begin{pmatrix} 0 & -U_\ell \\ -U_\ell^\dagger & 0 \end{pmatrix} + \left(\begin{array}{c} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ \hline \end{array} \right)_{k-1,r}^{k,r} + \left(\begin{array}{c} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ \hline \end{array} \right)_{k+1,r}^{k,r}.$$

The last two terms are meant, as before, to replace the terms $|110\rangle\langle 110|^c$ and $|100\rangle\langle 100|^c$. Once again, to uniquely identify the current clock state, we need to consider particles on top of each other. The first term represents a Hamiltonian that acts on the two particles in positions (k, r) and $(k + 1, r)$. These particles span a 36-dimensional space. The matrix shown above is in fact the restriction of this Hamiltonian to the eight-dimensional space spanned by

$$\begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \\ \hline |\uparrow\uparrow\rangle & |\downarrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\downarrow\rangle \end{array}$$

(recall that U_ℓ acts on two qubits and is therefore a 4×4 matrix). Everywhere else in this 36-dimensional subspace, this Hamiltonian acts trivially, i.e., is 0.

For the case $k = n$ we slightly modify the terms that identify the clock states

$$H''_{2nr+n} := \begin{pmatrix} 0 & -U_{2nr+n} \\ -U_{2nr+n}^\dagger & 0 \end{pmatrix} + \left(\begin{array}{c} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| \\ \hline \end{array} \right)_{n,r}^{n-1,r} + \left(\begin{array}{c} |\downarrow\uparrow\rangle\langle\downarrow\uparrow| \\ \hline \end{array} \right)_{n,r}.$$

For the case $k = 1$ we have

$$H''_{2nr+1} := \begin{pmatrix} 0 & -U_{2nr+1} \\ -U_{2nr+1}^\dagger & 0 \end{pmatrix} + (|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)_{1,r} + \left(\begin{array}{c} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow| \\ \hline \end{array} \right)_{2,r}^{1,r},$$

where the first term shows the restriction an operator acting on the particle $(1, r)$ to the four-dimensional space spanned by $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$ (recall that U_{2nr+1} is a 1-qubit gate).

5.5. Spectral Gap. The analysis of the spectral gap follows almost immediately from that in subsection 4.2.2. The main effort is in verifying that the restriction of each of our Hamiltonians to \mathcal{S} is identical to the restriction of the corresponding Hamiltonian in previous sections to \mathcal{S} , when both are constructed according to the modified quantum circuit of subsection 5.1. This, in fact, does not hold for H''_{input} , whose projection is not quite the same as that of H_{input} ; still, it is similar enough for the analysis in subsection 4.2.2 to hold.

CLAIM 5.3. $H''_{\mathcal{S},\text{clockinit}} = H_{\mathcal{S},\text{clockinit}}$.

Proof. Both Hamiltonians are diagonal in the basis $|\gamma_\ell^j\rangle$ with eigenvalue 0 for $\ell = 0$ and eigenvalue 1 for any $\ell > 0$. \square

CLAIM 5.4. For any $1 \leq \ell \leq L$, $H''_{\mathcal{S},\ell} = H_{\mathcal{S},\ell}$.

Proof. It is straightforward to verify that both Hamiltonians, when restricted to \mathcal{S} , are equal to

$$\sum_{j=0}^{2^n-1} [|\gamma_\ell^j\rangle\langle\gamma_\ell^j| + |\gamma_{\ell-1}^j\rangle\langle\gamma_{\ell-1}^j| - |\gamma_\ell^j\rangle\langle\gamma_{\ell-1}^j| - |\gamma_{\ell-1}^j\rangle\langle\gamma_\ell^j|]. \quad \square$$

For H''_{input} the situation is similar, although in this case the restriction to \mathcal{S} is not exactly the same. Still, the resemblance is enough for the same analysis to hold.

CLAIM 5.5. *Both $H_{\mathcal{S},\text{input}}$ and $H''_{\mathcal{S},\text{input}}$ are diagonal in the basis $|\gamma_\ell^j\rangle$. Moreover, the eigenvalue in both Hamiltonians corresponding to $|\gamma_\ell^j\rangle$ for $\ell = 0$ is exactly the number of 1's in the binary representation of j .*

Proof. The claim is easy to verify. \square

The similarity between the two Hamiltonians breaks down as follows. While the eigenvalues corresponding to $|\gamma_\ell^j\rangle$ for $\ell > 0$ are 0 in $H_{\mathcal{S},\text{input}}$, those in $H''_{\mathcal{S},\text{input}}$ might be positive (namely, for $0 \leq \ell \leq n$, the eigenvalue of $|\gamma_\ell^j\rangle$ is the number of 1's in the last $n - \ell$ digits in the binary representation of j). Nevertheless, due to the remark at the end of subsection 4.1, Lemma 4.2 holds here as well. We then get the following lemma.

LEMMA 5.6. *For any $0 \leq s \leq 1$, $H''_{\mathcal{S}}(s)$ has a spectral gap of $\Omega(L^{-3})$. Moreover, the ground state of $H''_{\mathcal{S},\text{final}}$ is $|\eta\rangle$.*

The rest of the proof of Theorem 1.3 is essentially the same as in subsection 4.2.2. By applying Lemma 4.7, we obtain the following lemma.

LEMMA 5.7. *For all $0 \leq s \leq 1$, $\Delta(H''(s)) = \Omega(L^{-3})$. Moreover, the ground state of $H''(1)$ is ϵ -close to $|\eta\rangle$.*

The proof is similar to that of Lemmas 4.8 and 4.9. This enables us to adiabatically generate the history state with exactly the same running time as in the 3-local case (when the number of gates is that of the modified circuit of subsection 5.1).

Finally, we would like to apply Lemma 3.10 as before. However, we cannot quite do this due to a technical issue: our Hilbert space is no longer a tensor product of computation qubits and clock qubits, and tracing out the clock qubits is meaningless. Nevertheless, a minor modification of that lemma still applies. We first add, say, L/ϵ identity gates to the end of the (modified) circuit. Now, the adiabatic computation produces a state close to the history state. We then measure the shape of the system without measuring the inner computational degrees of freedom. Due to the additional identity gates, with all but ϵ probability, the outcome of the measurement is a shape ℓ for $\ell \geq L$. If this is the case, then the state of the system is such that the active particles are in the final state of the circuit, as desired. This completes the proof of Theorem 5.1.

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