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Week 2 Problems (CH.3 : Noise)

3.1 (a) Derive (3.16) from binomial distribution & Stirling's approx.

$N = np$

(3.16) $p(x) = \dfrac{e^{-N} N^x}{x!}$ (bin. dist.) $p_n(x) = \binom{n}{x} p^x (1-p)^{n-x}$

(Stirling's) $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$, $\ln n! \approx n\ln(n) - n$ } n is large $\to \infty$, p is small $\to 0$

$p_n(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$\binom{n}{x} = \dfrac{n!}{(n-x)!\,x!}$

$p_n(x) = \binom{n}{x}\left(\dfrac{N}{n}\right)^x \left(1-\dfrac{N}{n}\right)^{n-x} = \dfrac{n!}{(n-x)!\,x!} \cdot \dfrac{N^x}{n^x}\left(1-\dfrac{N}{n}\right)^{n-x}$

$= \dfrac{(n-x+1)!}{x!\,n^x} N^x \left(1-\dfrac{N}{n}\right)^{n-x} = \dfrac{1}{x!}\underbrace{\left[\dfrac{n-x+1}{n}\cdot\dfrac{n-x}{n}\cdot\dfrac{n-x-1}{n}\cdots\dfrac{n}{n}\right]}_{①} N^x \left(1-\dfrac{N}{n}\right)^{n-x}$ ② ③

① $\lim\limits_{n\to\infty}\left[\dfrac{n-x+1}{n}\cdot\dfrac{n-x}{n}\,\dfrac{n-x-1}{n}\cdots\dfrac{n}{n}\right] = 1$

② $\lim\limits_{n\to\infty}\left(1-\dfrac{N}{n}\right)^n = \lim\limits_{-y\to\infty}\left(1+\dfrac{1}{y}\right)^{-yN} = e^{N}$ $\left[\text{for } y = -\dfrac{n}{N} = -\dfrac{1}{p},\ n = -yN\right]$ $\left[e = \lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^y\right]$

③ $\lim\limits_{n\to\infty}\left(1-\dfrac{N}{n}\right)^{-x} = 1^{-x} = 1$

$p_n(x) = \dfrac{N^x}{x!} e^{-N} = \dfrac{e^{-N} N^x}{x!}$ #

$\dfrac{x-m+1}{2} =$

(b) Derive (3.18) (c) (3.19)

$\langle (x-m+1)! \rangle = \langle (x-m+1)(x-m)(x-m-1)\ldots(x-1)(x)\rangle$ with m elements. $= N^m$

$= \left\langle \dfrac{x!}{(x-m)!} \right\rangle = \dfrac{(x)(x-1)(x-2)\cdots 1}{(x-m)(x-m-1)(x)}$

Recall: $e^x = \lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$

$e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

$= \dfrac{x}{x}\ \dfrac{x-1}{x+1}\ \dfrac{x-2}{x+2}\cdots$

$= \sum\limits_{x=0}^{\infty} \dfrac{e^{-N} N^x}{x!}(x-m+1)!$

3.2 | Photons gen. indep. & random @ N /sec. (avg).

rate within 1%? $\frac{1}{\sqrt{N}} = 0.01 \Rightarrow N = 10^4$ photons/s

within 1ppm? $\frac{1}{\sqrt{N}} = 10^{-6} \Rightarrow N = 10^{12}$ photons/s

Watts, visible light? $E = h\frac{c}{\lambda} = \frac{(6.626 \times 10^{-34} Js)(3 \times 10^8 \frac{m}{s})}{(380 \text{ to } 740 \text{ nm})} = 5.2 \times 10^{-19} J$ to $2.7 \times 10^{-19} J/\text{photon}$

$\|$ 1% : 5.2×10^{-15} W to 2.7×10^{-15} W

1ppm : 5.2×10^{-7} W to 2.7×10^{-7} W

3.3 | 20 kHz BW

(a) voltage source @ RT w/ $R_L = 10 k\Omega$.

input voltage for SNR : Johnson = 20dB?

fluctuating voltage. $\langle V^2_{\substack{input\\noise}} \rangle = 4kTR\Delta f = 3.2 \times 10^{-12}$

$V^2_{\substack{in\\noise}} \rightarrow V_{\substack{RMS\\noise}} \rightarrow V_{\substack{RMS\\signal}}$

$k_B = 1.38 \times 10^{-23}$ $\Downarrow$

$SNR = 10 \log_{10} \frac{\langle V^2_{sig} \rangle}{\langle V^2_{noise} \rangle} = 20dB$

$T = 293 K$ $\langle V_{\substack{rms\\noise}} \rangle = 1.8 \times 10^{-6}$

$R = 10 k\Omega$

$2 = \log_{10} \frac{V_{sig}}{V_{noise}}$

$\Delta f = 20 kHz$ $\| \langle V_{\substack{rms\\sig}} \rangle = 1.8 \times 10^{-4}$ V

$V_{noise} 10^2 = V_{sig}$

(b) capacitor to match voltage fluctuations?

$\frac{1}{2} C \langle V \rangle^2 = \frac{1}{2} k_B T$

$C = \frac{k_B T}{\langle V \rangle^2_{\substack{noise\\in}}} = 1.25 \times 10^{-9} = 1.25$ nF

(c) current source : RMS shot noise ~~when~~ for 1% of current?

$\langle I^2_{noise} \rangle = 2|q| \langle I \rangle_{sig} \Delta f$

$= + 6.4 \times 10^{-15} \langle I_{sig} \rangle$

$\langle I_{\substack{rms\\noise}} \rangle = 8 \times 10^{-8} \langle I_{sig} \rangle^{1/2}$

$0.01 = \frac{\langle I_{noise} \rangle}{\langle I_{sig} \rangle} = \frac{8 \times 10^{-8} \cancel{\langle I_{sig} \rangle^{1/2}}}{\langle I_{sig} \rangle^{1/2}}$

$\Downarrow \langle I_{sig} \rangle = 6.4 \times 10^{-11}$ A

3.4) stochastic process $x(t) \in [0,1]$

$\alpha\, dt$: $P(x)$ from 0 to 1 during dt.
$\beta\, dt$: $P(x)$ from 1 to 0 during dt. $\Big\rangle\ p_0(t) + p_1(t) = 1$

(a) Matrix diff. eqn : $\frac{dp_0(t)}{dt}$ & $\frac{dp_1(t)}{dt}$.

$$\frac{d}{dt}\begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = \underbrace{\begin{pmatrix} -\alpha & \beta \\ \alpha & -\beta \end{pmatrix}}_{\bar{A}} \begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} \qquad \overset{\alpha \to 1}{(-\alpha)} p_0(t) + (\overset{\beta \to 0}{\beta}) p_1(t) = \frac{d}{dt} p_0(t)$$

(b) diagonalize.

$$0 = \det(\bar{A} - \lambda I) = \det \begin{vmatrix} -\alpha-\lambda & \beta \\ \alpha & -\beta-\lambda \end{vmatrix} = (\alpha+\lambda)(\beta+\lambda) - \alpha\beta$$
$$= \alpha\beta + \alpha\lambda + \beta\lambda + \lambda^2 - \alpha\beta$$
$$0 = \alpha\lambda + \beta\lambda + \lambda^2$$
$$-\lambda^2 = (\alpha+\beta)\lambda$$
$$\lambda = -\alpha-\beta \quad ; \lambda = 0$$

For $\lambda = -\alpha-\beta$: $\begin{bmatrix} \beta & \beta \\ \alpha & \alpha \end{bmatrix}\begin{bmatrix} p_0(t) \\ p_1(t) \end{bmatrix} = 0$

$2\beta\, p_0(t) = 0$
$2\alpha\, p_1(t) = 0$ $\rightarrow \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

$\tilde{A} = \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & 0 \\ 0 & \frac{\beta}{\alpha+\beta} \end{bmatrix}$ For $\lambda = 0$: $\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$ $X = P z(t) = \begin{bmatrix} c_2 \beta e^{(-\alpha-\beta)t} \\ c_2 \alpha e^{(-\alpha-\beta)t} \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$p(t) = A e^{0t}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} + B e^{-(\alpha+\beta)t}\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = A\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} + B\begin{pmatrix} \beta \\ \alpha \end{pmatrix}$ $\quad 0 = A\alpha + B\beta \Rightarrow A = -B\frac{\beta}{\alpha} = \frac{\beta}{\alpha}\ \frac{1+A\beta}{\alpha}$

$1 = -A\beta + B\alpha \Rightarrow B = \frac{1 + A\beta}{\alpha}$

$\frac{d\vec{p}}{dt} = \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\alpha-\beta \end{bmatrix}}^{D} \vec{p}_d(t)$ $\qquad (-\alpha-\beta) p_2(t) = \frac{dp_2}{dt}$ $\qquad p = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$\vec{X}' = PDP^{-1}X$ $\qquad z_1' = 0 = c_1 e^0 \Rightarrow 0 = z_1$ $\qquad Pz = X$

$\vec{z}' = Dz$ $\qquad z_2' = (-\alpha-\beta) z_2 \Rightarrow c_2 e^{(-\alpha-\beta)t} = z_2$ $\qquad X = P(t)\begin{pmatrix} 0 \\ c_2 e^{(-\alpha-\beta)t} \end{pmatrix}$