HW #3

4.1 Verify entropy function satisfies: $H(p) = -\sum_{i=1}^{X} p_i \log(p_i)$

a) continuity

$p_i \in [0,1]$

for $p_i = 0$, $H(p) = 0$.

for $p_i = (0,1]$, $H(p) \in \mathbb{R}$.

it is continuous for its domain.

- p_i is continuous $[0,1]$
- $\log(p_i)$ is continuous $(0,1]$.

b) non-negativity. $0 \le p \le 1 \rightarrow \log p \le 0 \rightarrow -p\log p \ge 0$

For $p_i = 0$, $H(p) = 0$.

For $p_i = 0.1$, $H(0.1) = -0.1 \log(0.1) = -0.1(-1)$

$= +0.1.$

$p_i = 1$, $H(1) = 1 \cdot 0 = 0.$

Since $p_i \in [0,1]$, and $\log p_i \in$ neg $\mathbb{R}$ for $(0,1)$, $H(p)$ will always be pos.

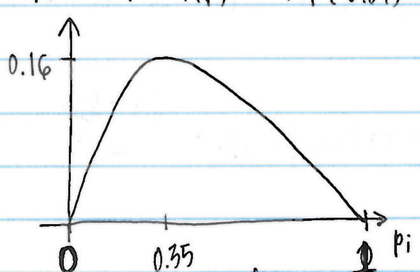
c) monotonicity

↳ output: always increasing or constant.

$p_i = 0 \rightarrow H(p) = 0.$

$p_i = 0.1 \rightarrow H(p) = -\sum 0.1(-1) = 0.1$

$p_i = 0.9 \rightarrow H(p) = -0.9(-0.04) = 0.04$



It is monotonic for increasing X value (summation sign).

d) independence.

4.2 $I(x,y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = H(x) - H(x|y)$

$= -\sum_x p(x) \log p(x) + \sum_x \sum_y p(x,y) \log p(x|y)$

$= +\sum_x \sum_y p(x,y) \left[-\log p(x) + \log p(x|y)\right]$

$= \sum_x \sum_y p(x,y) \log \frac{p(x|y)}{p(x)}$ $\qquad p(x|y) = \frac{p(x,y)}{p(y)}$

$= \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$

4.3. Binary channel, small prob ϵ error.

(a) prob. of error if bit sent indep. 3x

Majority voting: 3x

independently — $(\epsilon^3)^3 = \epsilon^9$.

(b) majority voting on majority voting.

$(\epsilon^9)^3 = \epsilon^{27}$

(c) mv on mv... on mv is done N times.

bits = 3N

error = $(\epsilon^m)^{3N}$, m : bits wanted

4.4. Differential entropy of Gaussian process.

$H(N) = \frac{1}{2}\log(2\pi e N)$ $\quad - \int_{-\infty}^{\infty} p(x) \ln N(x)\, dx$

$N(x) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-(x-\mu_N)^2/2\sigma_N^2}$

$H(N) = -\int_{-\infty}^{\infty} p(x) \log[N(x)]\, dx$

$H(x) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$

$= -\int_{-\infty}^{\infty} p(x)\left[\ln\frac{1}{\sqrt{2\pi\sigma_N^2}} + \frac{-(x-\mu_N)^2}{2\sigma_N^2}\right] dx$

$= -\int_{-\infty}^{\infty} p(x)\left[-\frac{1}{2}\ln(2\pi\sigma_N^2) - \frac{(x-\mu_N)^2}{2\sigma_N^2}\right] dx = \langle +\frac{1}{2}\ln(2\pi\sigma_N^2)\rangle + \frac{\langle(x-\mu_N)^2\rangle}{2\sigma_N^2}$ $\quad \langle x \rangle = \int x\, p(x)\, dx$

$\sigma^2 = \langle(x-\langle x\rangle)^2\rangle$

$= +\frac{1}{2}\ln(2\pi\sigma_N^2) + \frac{1}{2}$

4.5 3300 Hz BW, SNR = 20 dB

(a) Capacity?

$C = \Delta f \log(1 + \frac{S}{N})$

$= (3300\text{ Hz})[\log(1 + 10)]$

$= 3436$ bits/s

$SNR = 20 \log_{10}\left(\frac{V_S}{V_N}\right)$

$10^{\frac{20}{20}} = \frac{S}{N}$

(b) $1 \times 10^9 = (3300\text{ Hz})(\log(1 + \frac{S}{N}))$

$\frac{S}{N} = 10^{91221} \approx 10^5\text{ dB}$

4.6) $(x_1, x_2, \dots x_n)$: σ^2, x_0 gaussian.

$f(x_1 \dots x_n) = n^{-1} \sum_{i=1}^{n} x_i$ ⇒ estimator for x_0

↳ unbiased ①

↳ Cramér-Rao lower bound ②

$$\sigma^2(f) \geq \frac{1}{J(\alpha)}$$

① unbiased.

$$\langle f(x_1 \dots x_N) \rangle = \alpha = \langle n^{-1} \sum_i^n x_i \rangle$$

$$= n^{-1} \sum_i^n \langle x_i \rangle = x_0$$

② $$\langle (f - x_0)^2 \rangle = \langle (n^{-1} \sum_i^n x_i - x_0)^2 \rangle =$$

$$= n^{-2} \sum_i^n \underbrace{\langle (x_i - x_0)^2 \rangle}_{\sigma^2}$$

$$= \frac{\sigma^2}{n^2} \times n = \frac{\sigma^2}{n}$$

Fischer Information.

$$J_n(x_0) = n\, J(x_0) = n \langle \left[\frac{\partial}{\partial x_0} \log p_\alpha(x_0) \right]^2 \rangle$$

$$= n \langle \left[\frac{\partial}{\partial x_0} \log \mathcal{N}(x_0, \sigma^2) \right]^2 \rangle$$

$$= n \langle (x_i - x_0)^2 \rangle$$

① = ②