$\boxed{4.1}$ $H(p) = -\sum_{i=1}^{X} p_i \log p_i$

a) Continuity :

p and $\log p$ are both continuous for $0 \leq p \leq 1$

The product of continuous functions is continuous, therefore $p \log p$ is continuous.

The sum of continuous functions is continuous, thus $\sum_i p_i \log p_i$ is continuous.

$\therefore$ $H(p)$ is continuous for $0 \leq p \leq 1$

b) Non-negativity :

Since $0 \leq p \leq 1$, $\log_2 p \leq 0$

↳ $p \geq 0$ and $\log p \leq 0$ s $p \log p \leq 0$

↳ $\sum_i p_i \log p_i \leq 0$

↳ $-\sum_i p_i \log p_i \geq 0$

$\therefore$ $H(p) \geq 0$

Since all $p \log p \leq 0$, $H(p) = 0$ only if all $p \log p = 0$

This happens when for all i, either $p = 0$ or $p = 1$ ($\log p = 0$).

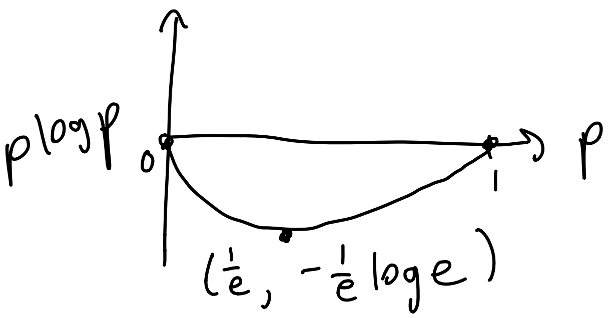
This means we are sure of the value x therefore only one $p_i = 1$ and all other $p_i = 0$.

c) Boundedness:

$$H(p) = -\sum_{i=1}^{X} p_i \log p_i$$

$$\frac{d}{dp}(p \cdot \log_2 p) = \log_2 p + \log_2 e = 0 \rightarrow p_{min} = \frac{1}{e}$$

$$\frac{1}{e} \cdot \log_2 \frac{1}{e} = -\frac{1}{e}\log_2 e$$



$$p \log p \geq -\frac{1}{e}\log e$$

$$\sum_{i=1}^{X} p_i \log p_i \geq X \cdot \left(-\frac{1}{e}\log e\right)$$

$$-\sum_{i=1}^{X} p_i \log p_i \leq X \cdot \frac{1}{e}\log e$$

$H(p) \leq \underbrace{X \cdot \frac{1}{e}\log e}_{C(X)}$ which is increasing w.r.t. X ✓

$H(p) = X \cdot \frac{1}{e}\log e$ for all $p_i = p_{min}$ so equally likely ✓

d) Independence:

$$H(p, q) = -\sum_{i=1}^{X} p_i \log p_i - \sum_{j=1}^{Y} q_j \log q_j = H(p) + H(q)$$

?

4.2 Mutual information

We see in eqn 4.9 that $H(x,y) = H(x|y) + H(y)$

Plug in: $H(x) + H(y) - H(x,y) = H(x) + H(y) - H(x|y) - H(y)$

$= H(x) - H(x|y)$ ✓

Equivalently by 4.9, $H(x,y) = H(y|x) + H(x)$

Plug in: $H(x) + H(y) - H(x,y) = H(x) + H(y) - H(y|x) - H(x)$

$= H(y) - H(y|x)$ ✓

$$H(x) = -\sum_x p(x) \log p(x)$$

$$H(y) = -\sum_y p(y) \log p(y)$$

$$H(x,y) = -\sum_x \sum_y p(x,y) \log p(x,y)$$

$H(x) + H(y) - H(x,y) =$

$$-\sum_x p(x) \log p(x) - \sum_y p(y) \log p(y) + \sum_x \sum_y p(x,y) \log p(x,y)$$

$$= \sum_x \sum_y p(x,y) \cdot \left(\log p(x,y) - \log p(x) - \log p(y) \right)$$

?

4.3 Binary channel w/ prob ϵ of bit error

a) Perror if bit sent 3 times + majority voting

$$P_{error} = \epsilon \cdot \epsilon \cdot (1-\epsilon) + \epsilon \cdot (1-\epsilon) \cdot \epsilon + (1-\epsilon) \cdot \epsilon \cdot \epsilon + \epsilon \cdot \epsilon \cdot \epsilon$$

$$= 3 \cdot \epsilon^2 (1-\epsilon) + \epsilon^3$$

$$= \epsilon^3 + 3\epsilon^2 - 3\epsilon^3$$

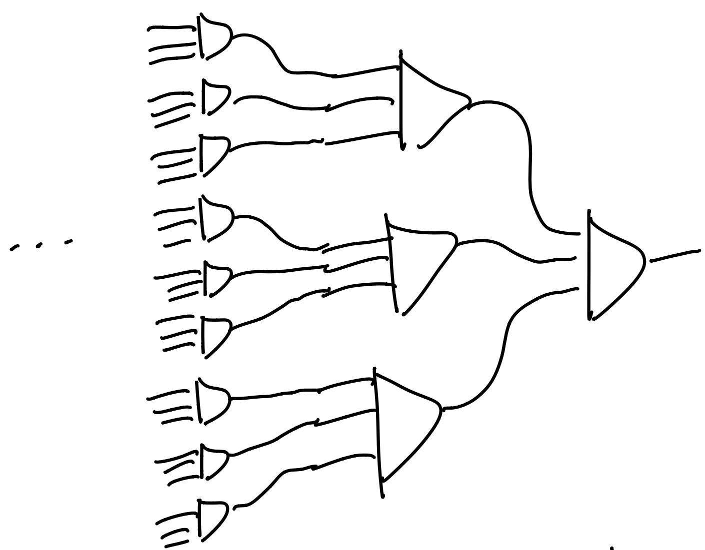
$$= \boxed{3\epsilon^2 - 2\epsilon^3}$$

b) $p_1 = 3\epsilon^2 - 2\epsilon^3$

$$p_2 = 3p_1^2 - 2p_1^3$$

$$= \boxed{3(3\epsilon^2 - 2\epsilon^3)^2 - 2(3\epsilon^2 - 2\epsilon^3)^3}$$

c)



$N=3$ $\quad$ $N=2$ $\quad$ $N=1$

bits $= 3^3$ $\quad$ bits $= 3^2$ $\quad$ bits $= 3^1$

$$\boxed{\begin{array}{l} \text{Bits} = 3^N \\ P_{error} \to 0 \text{ as } N \to \infty \\ P_{error} \sim \epsilon^{2N} \end{array}}$$

$\boxed{4.4}$ Differential entropy: $H(x) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-(x-\mu_N)^2/2\sigma_N^2}$$

$$H(\mathcal{N}) = -\int_{-\infty}^{\infty} \mathcal{N}(x) \ln \mathcal{N}(x)\, dx$$

$$= -\frac{1}{\sqrt{2\pi\sigma_N^2}} \int_{-\infty}^{\infty} e^{-(x-\mu_N)^2/2\sigma_N^2} \ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-(x-\mu_N)^2/2\sigma_N^2}\right) dx$$

$$= -\frac{1}{\sqrt{2\pi\sigma_N^2}} \int_{-\infty}^{\infty} e^{-(x-\mu_N)^2/2\sigma_N^2} \left(\ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right) - \frac{(x-\mu_N)^2}{2\sigma_N^2}\right) dx$$

$$= -\frac{1}{\sqrt{2\pi\sigma_N^2}} \Bigg[\ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right) \int_{-\infty}^{\infty} e^{-(x-\mu_N)^2/2\sigma_N^2} dx$$

$$- \frac{1}{2\sigma_N^2} \int_{-\infty}^{\infty} (x-\mu_N)^2 e^{-(x-\mu_N)^2/2\sigma_N^2} dx \Bigg]$$

$$= -\frac{1}{\sqrt{2\pi\sigma_N^2}} \left[\ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right) \cdot \sigma_N\sqrt{2\pi} - \frac{1}{2\cancel{\sigma_N^2}} \cdot \sigma_N^3 \sqrt{2\pi} \right]$$

$$= -\frac{1}{\cancel{\sigma_N\sqrt{2\pi}}} \left[\ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right) \cdot \cancel{\sigma_N\sqrt{2\pi}} - \frac{1}{2} \cdot \cancel{\sigma_N\sqrt{2\pi}} \right]$$

$$= -\left[\ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right) - \frac{1}{2} \right]$$

$$= \frac{1}{2} - \ln\left(\frac{1}{\sqrt{2\pi\sigma_N^2}}\right)$$

$$= \frac{1}{2} + \ln\left(\sqrt{2\pi\sigma_N^2}\right)$$

$$= \ln(e^{1/2}) + \ln\left(\sqrt{2\pi\sigma_N^2}\right)$$

$$= \ln\left(\sqrt{2\pi e\sigma_N^2}\right)$$

$$= \frac{1}{2} \ln(2\pi e\sigma_N^2)$$

$$= \boxed{\frac{1}{2} \ln(2\pi e N) \quad \text{where } N = \sigma_N^2}$$

✓

$\boxed{4.5}$ Bandwidth $= 3300\,\text{Hz}$

SNR $= 20\,\text{dB}$

a) Capacity: (eqn 4.28)

$$C = \Delta f \cdot \frac{1}{2}\log_2\left(1 + \frac{S}{N_0}\cdot\frac{1}{\Delta f}\right)$$

$$10\log_{10}\left(\frac{S}{N_0}\right) = 20\,\text{dB} \rightarrow \frac{S}{N_0} = 100$$

$$C = \boxed{71.1 \text{ bits/second}}$$

b)
$$10^9 = 3300\cdot\frac{1}{2}\log_2\left(1+\frac{S}{N_0}\cdot\frac{1}{3300}\right)$$

$$6.06\times10^5 = \log_2\left(1+\frac{S}{N_0}\cdot\frac{1}{3300}\right)$$

$$1+\frac{S}{N_0}\cdot\frac{1}{3300} = 2^{6.06\times10^5}$$

$$\frac{S}{N_0} = 3300\left(2^{6.06\times10^5}-1\right)$$

$$\text{SNR} = 10\log_{10}\left(3300\left(2^{6.06\times10^5}\cancel{-1}\right)\right)$$

$$= 10\left(\log_{10}(3300) + \log_{10}\left(2^{6.06\times10^5}\right)\right)$$

$$= 10\left(\log_{10}(3300) + \log_{10}\left(\left(2^{10}\right)^{6.06\times10^4}\right)\right)$$

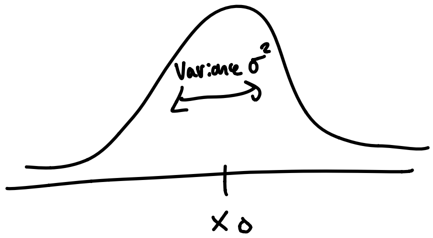
$$\approx 10\left(\log_{10}(3300) + \log_{10}\left(\left(10^{3}\right)^{6.06\times10^4}\right)\right)$$

$$\approx 10\left(\log_{10}(3300) + \log_{10}\left(10^{1.82\times10^5}\right)\right)$$

$$\approx 10\left(3.52 + 1.82\times10^5\right)$$

$$\approx \boxed{1.82\times10^6\,\text{dB}}$$

$\boxed{4.6}$



Draw $x_1, x_2, \dots, x_n$ from dist.

$f(x_1, \dots, x_n) = n^{-1} \sum_{i=1}^{n} x_i$ unbiased estimator for x_0 that achieves Cramér-Rao lower bound

Unbiased : $\langle f(x_1, \dots, x_n) \rangle = x_0$

↳ by inspection, since f is the mean of the measurements, and the expected value of the mean of measurements is the mean of the distribution ✓

$$p_{x_0}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-x_0)^2/2\sigma^2}$$

$$J(x_0) = \int_{-\infty}^{\infty} \frac{1}{p_{x_0}(x)} \left[\frac{\partial p_{x_0}(x)}{\partial x_0} \right]^2 dx$$

$$= \frac{\sqrt{2\pi\sigma^2}}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{(x-x_0)^2/2\sigma^2} \left[\frac{\partial}{\partial x_0} e^{-(x-x_0)^2/2\sigma^2} \right]^2 dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{(x-x_0)^2/2\sigma^2} \left[\frac{x-x_0}{\sigma^2} e^{-(x-x_0)^2/2\sigma^2} \right]^2 dx$$

$$= \frac{1}{\sigma^3\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-x_0)^2/2\sigma^2} dx$$

$$= \frac{1}{\sigma^3\sqrt{2\pi}} \cdot \sigma\sqrt{2\pi}$$

$$= \frac{1}{\sigma^2}$$

Cramer-Rao : $\sigma^2(f) \geq \frac{1}{J(x_0)}$

$\sigma^2(f) \geq \sigma^2$ ✓