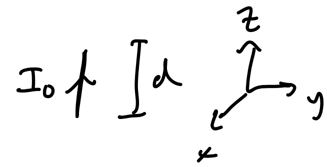
$\boxed{8.1}$ $\vec{E} = \frac{1}{i\omega\mu_0\epsilon_0}\nabla(\nabla\cdot\vec{A}) - i\omega\vec{A}$



$$\vec{A} = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r}\cos\theta\cdot\hat{r} - \mu_0\frac{I_0 d e^{-ikr}}{4\pi r}\sin\theta\cdot\hat{\theta}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \cancel{\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mu_0\frac{I_0 d e^{-ikr}}{4\pi r}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\mu_0\frac{I_0 d e^{-ikr}}{4\pi r}\sin\theta\cdot\sin\theta\right)$$

$$= \frac{\mu_0 I_0 d\cos\theta}{4\pi r^2}\frac{\partial}{\partial r}\left(re^{-ikr}\right) - \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)$$

$$= \frac{\mu_0 I_0 d\cos\theta}{4\pi r^2}e^{-ikr}(-ikr+1) - \frac{\mu_0 I_0 d e^{-ikr}}{4\pi r^2\cancel{\sin\theta}}2\cancel{\sin\theta}\cos\theta$$

$$= \frac{\mu_0 I_0 d}{4\pi r^2}e^{-ikr}\cos\theta\,(1-ikr-2)$$

$$= \frac{\mu_0 I_0 d}{4\pi r^2}e^{-ikr}\cos\theta\,(-ikr-1)$$

$$\nabla(\nabla\cdot\vec{A}) = \nabla\left[\frac{\mu_0 I_0 d}{4\pi r^2}e^{-ikr}\cos\theta\,(-ikr-1)\right]$$

$$= \frac{\mu_0 I_0 d}{4\pi}\left[\cos\theta\frac{\partial}{\partial r}\underbrace{\left(\frac{1}{r^2}e^{-ikr}(-ikr-1)\right)}_{-\frac{ik}{r}e^{-ikr}-\frac{1}{r^2}e^{-ikr}}\hat{r} + \underbrace{\frac{1}{r}\cdot\frac{1}{r^2}e^{-ikr}(-ikr-1)\frac{\partial}{\partial\theta}(\cos\theta)\,\hat{\theta}}_{-e^{-ikr}\left(-\frac{ik}{r^2}-\frac{1}{r^3}\right)\sin\theta\cdot\hat{\theta}}\right]$$

$$= \frac{\mu_0 I_0 d}{4\pi}\left[\cos\theta e^{-ikr}\left(-\frac{k^2}{r}+2\frac{ik}{r^2}+\frac{2}{r^3}\right)\hat{r} - e^{-ikr}\left(-\frac{ik}{r^2}-\frac{1}{r^3}\right)\sin\theta\cdot\hat{\theta}\right]$$

$$= \frac{\mu_0 I_0 d}{4\pi r}\left[\cos\theta e^{-ikr}\left(-k^2+2\frac{ik}{r}+\frac{2}{r^2}\right)\hat{r} - e^{-ikr}\left(-\frac{ik}{r}-\frac{1}{r^2}\right)\sin\theta\cdot\hat{\theta}\right]$$

$$E_\theta = \frac{+1}{i\omega\cancel{\mu_0}\epsilon_0}\frac{\cancel{\mu_0}I_0 d}{4\pi r}e^{-ikr}\left(+\frac{ik}{r}+\frac{1}{r^2}\right)\sin\theta + i\omega\mu_0\frac{I_0 d e^{-ikr}}{4\pi r}\sin\theta$$

$$= \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{1}{i\omega\epsilon_0 r}\left(\frac{ik}{r}+\frac{1}{r^2}\right) + i\omega\mu_0\frac{1}{r}\right)\sin\theta$$

$$= \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{k}{\omega\epsilon_0 r^2} + \frac{1}{i\omega\epsilon_0 r^3} + \frac{i\omega\mu_0}{r}\right)\sin\theta$$

where $k = \frac{\omega}{c} = \omega\sqrt{\epsilon_0\mu_0}$

$$= \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{i\omega\mu_0}{r} + \frac{\sqrt{\epsilon_0\mu_0}}{\epsilon_0 r^2} + \frac{1}{i\omega\epsilon_0 r^3}\right)\sin\theta$$

$$= \boxed{\frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{i\omega\mu_0}{r} + \frac{1}{r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{1}{i\omega\epsilon_0 r^3}\right)\sin\theta}$$

$$E_r = \frac{1}{i\omega\cancel{\mu_0}\epsilon_0}\frac{\cancel{\mu_0}I_0 d}{4\pi r}\cos\theta e^{-ikr}\left(-k^2+2\frac{ik}{r}+\frac{2}{r^2}\right) - i\omega\mu_0\frac{I_0 d e^{-ikr}}{4\pi r}\cos\theta$$

$$= \frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{1}{i\omega\epsilon_0 r}\left(-\omega^2\epsilon_0\mu_0 + 2\frac{i}{r}\omega\sqrt{\epsilon_0\mu_0} + \frac{2}{r^2}\right) - i\omega\mu_0\frac{1}{r}\right)\cos\theta$$

$$= \frac{I_0 d}{4\pi}e^{-ikr}\left(\cancel{\frac{i\omega\mu_0}{r}} + \frac{2}{r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{2}{i\omega\epsilon_0 r^3} - \cancel{i\omega\mu_0\frac{1}{r}}\right)\cos\theta$$

$$= \boxed{\frac{I_0 d}{4\pi}e^{-ikr}\left(\frac{2}{r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} + \frac{2}{i\omega\epsilon_0 r^3}\right)\cos\theta}$$

$\boxed{8.2}$

Assume $\lambda \ll 1\,\text{km}$
(working in far field)

$$\langle \vec{p} \rangle = \hat{r}\, \frac{I_0^2 k^2 d^2}{32\pi^2 r^2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2\theta$$

Take $\sin\theta = 1$

$$|\vec{p}| = \frac{I_0^2 k^2 d^2}{32\pi^2 r^2}$$

$$W = \frac{I_0^2 k^2 d^2}{12\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} = 1\,\text{kW} \rightarrow I_0^2 k^2 d^2 = 12\pi \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot 1\,\text{kW}$$

$$|\vec{p}| = \frac{12\cancel{\pi}}{32\pi^{\cancel{2}}} \frac{1}{r^2} \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot 1\,\text{kW}$$

$$= \frac{3}{8\pi} \left(\frac{1}{377\,\Omega}\right) \frac{1\,\text{kW}}{(1\,\text{km})^2}$$

$$= \boxed{3.17 \times 10^{-7}\ \frac{\text{W}}{\Omega\,\text{m}^2}}$$

$$\vec{E} \approx \frac{I_0 d}{4\pi} e^{-ikr} \left(\frac{i\omega\mu_0}{r}\right) \text{ in far field}$$

$$|\vec{E}| = \frac{I_0 d\, \omega \mu_0}{4\pi r}$$

$$|\vec{E}|^2 = \frac{I_0^2 d^2 \omega^2 \mu_0^2}{16\pi^2 r^2}$$

$$I_0^2 k^2 d^2 = I_0^2 \omega^2 d^2 \epsilon_0 \mu_0 = 12\pi \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot 1\,\text{kW}$$

$$\hookrightarrow I_0^2 \omega^2 d^2 = 12\pi \frac{1}{\epsilon_0 \mu_0} \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot 1\,\text{kW}$$

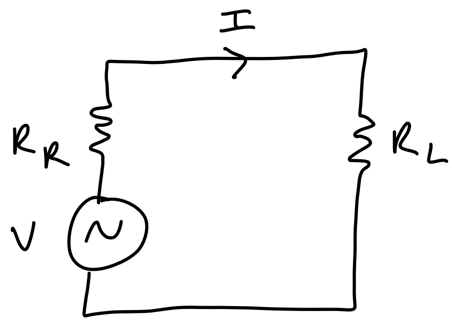
$$|\vec{E}|^2 = \frac{12\pi}{16\pi^2} \frac{\mu_0^{\cancel{2}}}{\epsilon_0 \cancel{\mu_0}} \sqrt{\frac{\epsilon_0}{\mu_0}} \cdot \frac{1}{r^2} \cdot 1\,\text{kW}$$

$$= \frac{3}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{(1\,\text{km})^2} \cdot 1\,\text{kW}$$

$$= .090\ \frac{\text{W}\,\Omega}{\text{m}^2}$$

$$|\vec{E}| = \boxed{0.300 \sqrt{\frac{\text{W}\,\Omega}{\text{m}^2}}}$$

$\boxed{8.3}$ R_{load} to maximize power



$$V_{pk} = I_{pk}(R_R + R_L)$$

$$I_{pk} = \frac{V_{pk}}{(R_R + R_L)}$$

$$V_{rms} = \frac{|V|}{\sqrt{2}}$$

$$I_{rms} = \frac{|V|}{\sqrt{2}(R_R + R_L)}$$

$$P = I_{rms}^2 \cdot R_L = \frac{|V|^2}{2(R_R+R_L)^2} \cdot R_L = \frac{1}{2} \cdot \frac{|V|^2 R_L}{R_R^2 + 2R_L R_R + R_L^2}$$

$$\frac{d}{dR_L} P = \frac{1}{2}|V|^2 \cdot \frac{R_R - R_L}{(R_R + R_L)^3} = 0$$

↳ assuming resistances > 0, $R_L = R_R$ to optimize

$$\frac{d^2}{dR_L^2} P \Big|_{R_L = R_R} = -\frac{1}{4R_R^3} < 0$$ hence we found a local max

$$\therefore P_{max} = P\Big|_{R_L = R_R} = \frac{1}{2} \frac{|V|^2 R_R}{4R_R^2} = \boxed{\frac{1}{8} \frac{|V|^2}{R_R}}$$

$\boxed{8.4}$ Gain & area of infinitesimal dipole

$$G = \max_{\theta, \phi} \frac{P(r=1, \theta, \phi)}{W/4\pi}$$

$$P = \frac{I_0^2 k^2 d^2}{32\pi^2 r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2\theta = \frac{I_0^2 \left(\frac{2\pi}{\lambda}\right)^2 d^2}{32\pi^2 r^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2\theta \quad \text{since } k = \frac{2\pi}{\lambda}$$

$$= \frac{I_0^2 d^2}{8 r^2 \lambda^2}\sqrt{\frac{\mu_0}{\epsilon_0}} \sin^2\theta$$

$$\max_{\theta, \phi} P(r=1) = \frac{I_0^2 d^2}{8\lambda^2}\sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$W = \frac{I_0^2 \pi}{3}\sqrt{\frac{\mu_0}{\epsilon_0}} \frac{d^2}{\lambda^2}$$

$$G = \frac{\frac{\cancel{I_0^2} \cancel{d^2}}{8\cancel{\lambda^2}}\cancel{\sqrt{\frac{\mu_0}{\epsilon_0}}} \cdot 4\cancel{\pi}}{\frac{\cancel{I_0^2 \pi}}{3}\cancel{\sqrt{\frac{\mu_0}{\epsilon_0}}}\frac{\cancel{d^2}}{\cancel{\lambda^2}}} = \frac{1/2}{1/3} = \boxed{\frac{3}{2}}$$

Power delivered to matched load: $\frac{|V|^2}{8 R_{Rad}}$

Area???