Quantum Computing and Communications Problem Set

Sara V. Fernandez
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16.1) Equation 16.33 is Ehrenfest's Theorem: $i\hbar \frac{d}{dt}\langle \hat{A} \rangle = \langle [\hat{A}, \hat{H}] \rangle$

If the observable (A) has a specific time dependence,

$i\hbar \frac{d}{dt}\langle \hat{A} \rangle = \langle [\hat{A}, \hat{H}] \rangle$ becomes (by knowing $\hbar = \frac{h}{2\pi}$ and $[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}$)

$\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{h}\langle [\hat{H}, \hat{A}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$

Equivalently: $= \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle$

usual quantum mechanical evolution from Hamiltonian ↳ explicit time dependence of $\hat{A}$

16.2) $\hat{\rho} = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ is $\hat{\rho}$ in its diagonal form & where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$

$\mathrm{Tr}(\hat{\rho}^2) = \mathrm{Tr}\left(\sum_{i,j} \lambda_i \lambda_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\right) = \sum_i \lambda_i^2$ given $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ (orthonormality)

Cauchy-Schwarz inequality for real numbers: The expectation of a product will be bounded by the product of the expectations.

$\sum_i \lambda_i^2 \leq \left(\sum_i \lambda_i\right)^2 = 1 \quad \therefore \mathrm{Tr}(\hat{\rho}^2) \leq 1$

↳ a measure of how pure the state is

16.3a) Pauli matrices from Equation 16.71: (in the basis of z operators)

$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Solve $\hat{\sigma}_x \vec{v} = \lambda \vec{v}$ where $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$:

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{cases} b = \lambda a \\ a = \lambda b \end{cases}$

Substituting a into b: $a = \lambda(\lambda a) \rightarrow \lambda^2 = 1 \;\therefore\; \lambda = \pm 1$

For $\lambda = 1$, $b = a \rightarrow \vec{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To normalize, $|\vec{v}| = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore \vec{v}_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda = -1$, $b = -a \rightarrow \vec{v}_- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

To normalize, $|\vec{v}| = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\therefore \vec{v}_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Finding the eigenvectors of $\hat{\sigma}_y$ in the same manner, except now

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{cases} -ib = \lambda a \\ ia = \lambda b \end{cases}$$

$\therefore$ For $\lambda = 1$, $\vec{v}_+ = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\vec{v}_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

For $\lambda = -1$, $\vec{v}_- = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\vec{v}_{norm} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

16.3b) singlet state: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$

$$\langle + |_{y_1} |\Psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|\downarrow\rangle_2 - \frac{i}{\sqrt{2}}|\uparrow\rangle_2\right) = \frac{1}{2}\left(|\downarrow\rangle_2 - i|\uparrow\rangle_2\right)$$

$\therefore$ The second spin is in:

$$|\phi_2\rangle = \frac{1}{2}\left(|\downarrow\rangle - i|\uparrow\rangle\right) = -i \cdot \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right) = -i|-\rangle_y$$

$$-i|-\rangle_{y_2}$$

Projecting spin 1 onto $m_{x_1} = \frac{1}{2} \rightarrow$ spin 2 is in $-|-\rangle_{x_2}$

Projecting spin 1 onto $m_{y_1} = \frac{1}{2} \rightarrow$ spin 2 is in $-i|-\rangle_{y_2}$

$\therefore m_{1x} = \frac{1}{2} \rightarrow m_{2x} = -\frac{1}{2}$ & $m_{1y} = \frac{1}{2} \rightarrow m_{2y} = \frac{1}{2}$

16.3c) $m_{1x} = -m_{2x}$ & $m_{1y} = -m_{2y}$

$\therefore$ $m_{1x} m_{2y} = -m_{1y} m_{2x}$ (perfect anti correlation of the singlet state when both spins are on the same axis)

16.3d) Using $\langle \psi | \hat{\sigma}_{1x} \hat{\sigma}_{2y} | \psi \rangle = -\langle \psi | \hat{\sigma}_{1y} \hat{\sigma}_{2x} | \psi \rangle$ from before:

$$\langle \psi | \hat{\sigma}_{1x} \hat{\sigma}_{2y} | \psi \rangle = -\langle \psi | \hat{\sigma}_{1y} \hat{\sigma}_{2x} | \psi \rangle$$

16.3e) $$\hat{\sigma}_{1x} \hat{\sigma}_{2y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\sigma}_{1y} \hat{\sigma}_{2x} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\sigma}_{1x} \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \hat{\sigma}_{2x} = \begin{bmatrix} 0 & 0 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \end{bmatrix}$$

Applying this to the singlet state:

Since $|\psi| = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ & $\langle \psi | = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$,

$$\langle \psi | \hat{\sigma}_{1x} \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \hat{\sigma}_{2x} | \psi \rangle = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

16.3f) These appear to be consistent. Part e sums the two terms from d. Part e says $\langle \hat{\sigma}_{1y} \hat{\sigma}_{2y} + \hat{\sigma}_{1y} \hat{\sigma}_{2x} \rangle = \langle \hat{\sigma}_{1x} \hat{\sigma}_{2y} \rangle + \langle \hat{\sigma}_{1y} \hat{\sigma}_{2x} \rangle$, d shows $\langle \hat{\sigma}_{1y} \hat{\sigma}_{2x} \rangle = -\langle \hat{\sigma}_{1y} \hat{\sigma}_{2x} \rangle$, and so their sum is zero $\langle \hat{\sigma}_{1x} \hat{\sigma}_{2y} \rangle + \langle \hat{\sigma}_{1y} \hat{\sigma}_{2x} \rangle = 0$

16.4) Equation 16.20 (spectral representation): $\hat{X} = \sum_n x_n |n\rangle\langle n|$

$$R_n(\theta) = e^{-i\frac{\theta}{2}\hat{\sigma}_n} = \sum_{n=\pm1} e^{-i\frac{\theta}{2}n} |n\rangle\langle n|$$

diagonal operator in eigenbasis of $\hat{\sigma}_n$ w/ eigenvalues $e^{-i\frac{\theta}{2}}$ & $e^{i\frac{\theta}{2}}$

Writing each $R_n(\theta)$ explicitly with Pauli matrices:

Remember $e^{-i\theta\hat{\sigma}_n/2} = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_n$ where $\hat{\sigma}_n^2 = I$

↳ true because $p^2 = I$ (Hermitian) → identity stands

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\sigma_x = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\sigma_y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\sigma_z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

16.5) NOT gates have a rotation of 180°: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ☆

$\therefore$ $\sqrt{NOT}$ gates have a rotation of 90°: $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

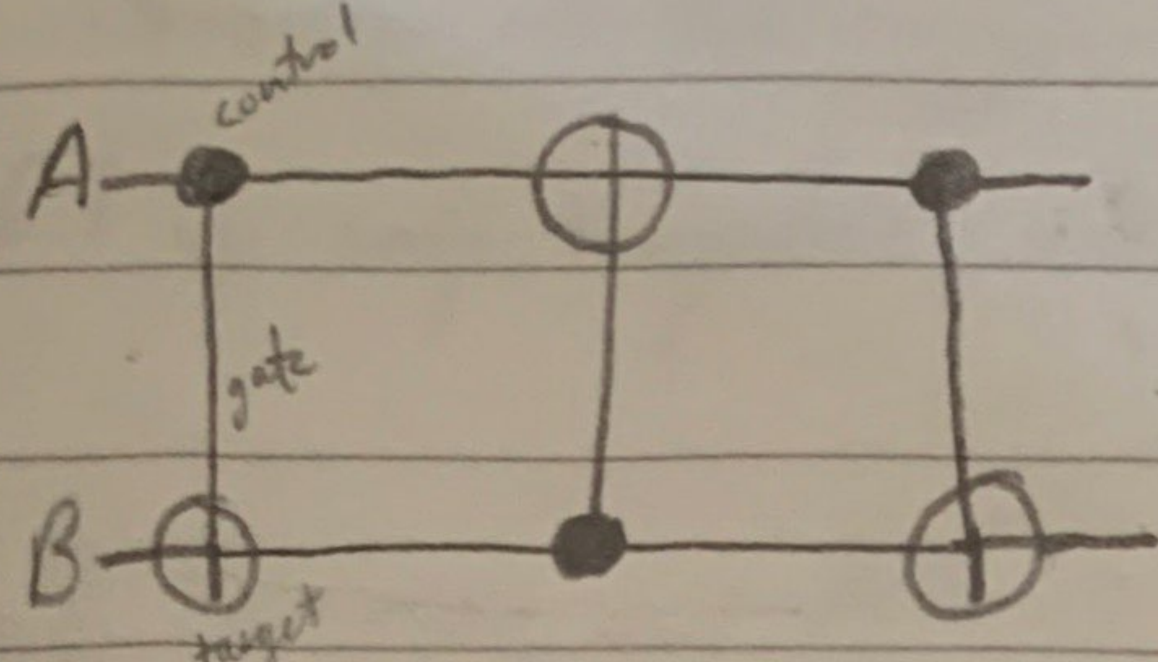
$$\sqrt{NOT}^2 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

(Hadamard transform Equation 16.108)

16.6) For N=1 qubit: $\hat{H}|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For N=2 qubits: $(\hat{H}\otimes\hat{H})|00\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(Hadamard transformations)

16.7) 

I think the gate swaps A&B due to quantum entanglement as is explained in Figure 16.2.

Refer to 16.109 matrix: CNOT $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, swap gates, and teleportation.

Useful to reference Figure 16.5: Circuit for a quantum Fourier transform

16.8a) The total number of gates (Hadamards and controlled phase rotations):

$$\sum_{k=1}^{N} (N-k) = \frac{N(N-1)}{2} + N = \frac{N(N+1)}{2} \text{ total gates}$$

$\therefore$ QFT requires $O(N^2)$ operations on N qubits.

(Confirmed on page 299) → $Z_{>} = O((\log N)^2)$ (better than FFT)

16.8b) $T = \frac{\Phi}{\omega_0}$ spin rotate rate

total phase shift

$$\Phi = \sum_{k=2}^{n} \frac{2\pi}{2^k} = 2\pi \frac{(\frac{1}{2})^2 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = \pi\left(1 - \left(\tfrac{1}{2}\right)^{n-1}\right)$$

$$T = \frac{\Phi}{\omega_0} = \frac{\pi}{\omega}\left(1 - \left(\tfrac{1}{2}\right)^{n-1}\right)$$ required for a Fourier transform

$\therefore O(N)$ meaning QFT does linear time (misleading to count operations)