Circuits, Transmission Lines, and Waveguides Problem Set

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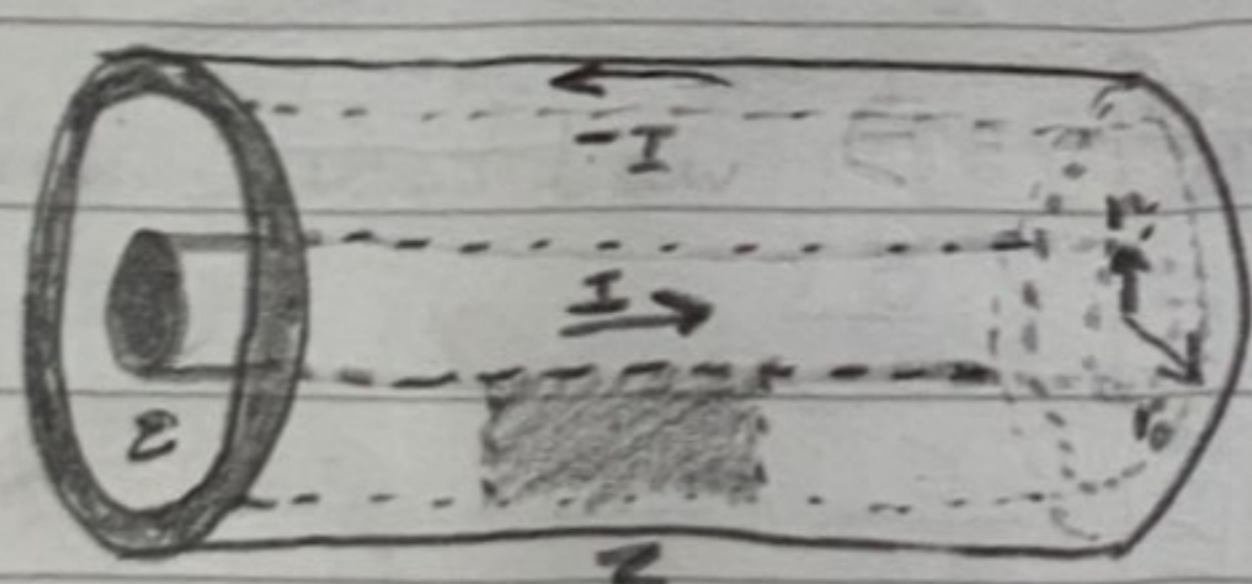
9.1) The twists act like a Faraday cage to reduce electromagnetic interference that is generated by the counterfactual — the case in which two parallel wires induce a magnetic field between them when the direction of current flowing through them is opposite. Cables are shielded to both reduce interference from external EM sources and RF sources as well as prevent internal EM from affecting external EM sensitive systems.

9.2) skin depth: $\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi \nu \mu \sigma}}$ where ν = frequency
μ = permeability
σ = conductivity

$$\delta = \frac{1}{\sqrt{\pi (10^4 Hz) \underbrace{(4\pi \cdot 10^{-7} \frac{H}{m})}_{\text{magnetic permeability of free space } \mu_0} (4 \frac{S}{m})}}$$

$$\delta = 2.52 mm$$

9.3) Integrate Poynting's vector $\vec{P} = \vec{E} \times \vec{H}$



$E(r_i < r < r_0) = \frac{V}{r \ln(\frac{r_0}{r_i})} \hat{r}$ (radial)

$H(r_i < r < r_0) = \frac{I}{2\pi r} \hat{\phi}$ (from Stokes' Law)

Solve for E in this case using Gauss's Law:

$\oint E \cdot dA \, \frac{Q_{enc}}{\varepsilon_0}$

$E(2\pi r L) = \frac{Q}{\varepsilon_0}$

$E = \frac{Q}{2\pi\varepsilon_0 r L}$

$\rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \leftarrow (\lambda = \frac{Q}{L})$

$V = \int_{r_i}^{r_0} E dr = \int_{r_i}^{r_0} \frac{\lambda}{2\pi\varepsilon r} dr$

$V = \frac{\lambda}{2\pi\varepsilon_0} \ln(\frac{r_0}{r_i})$

$\rightarrow \lambda = \frac{2\pi\varepsilon_0 V}{\ln(\frac{r_0}{r_i})}$

$\downarrow$

$E = \frac{V}{r \ln(\frac{r_0}{r_i})}$

Integrate $\vec{P} = \vec{E} \times \vec{H}$

$$P = \frac{V}{r\ln\left(\frac{r_2}{r_1}\right)}\hat{r} \times \frac{I}{2\pi r}\hat{\phi}$$

$$P = \frac{VI}{2\pi r^2 \ln\left(\frac{r_2}{r_1}\right)}\hat{z} \quad \text{(axial)}$$

$$P = \iint_{axial} P \cdot dA$$

$$P = \int_{r_1}^{r_2} \frac{VI}{2\pi r^2 \ln\left(\frac{r_2}{r_1}\right)}(2\pi r\, dr)$$

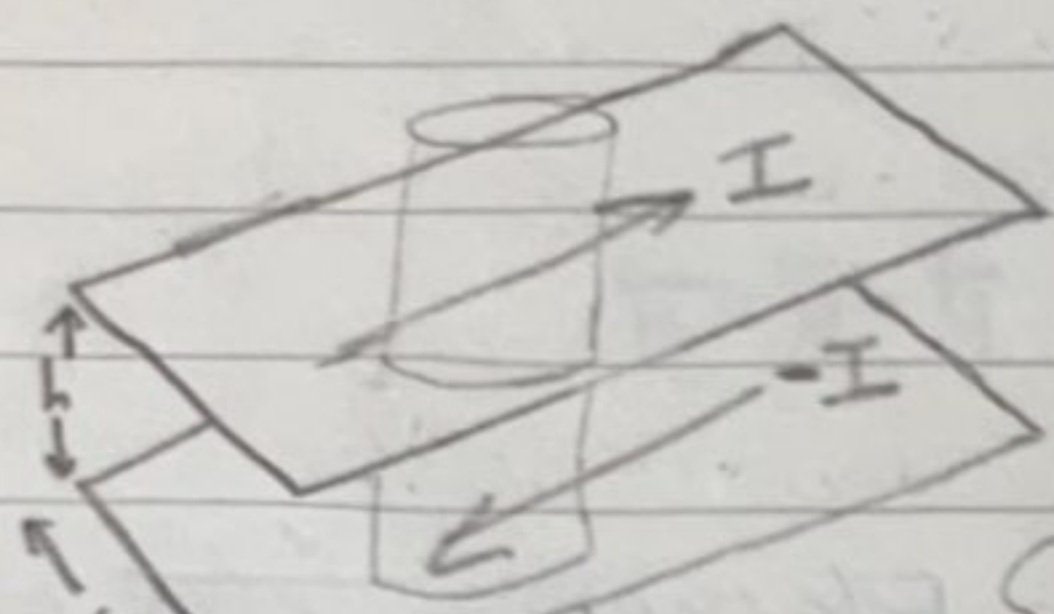
$$P = \frac{VI}{\ln\left(\frac{r_2}{r_1}\right)}\int_{r_1}^{r_2}\frac{dr}{r}$$

$$P = \frac{VI}{\ln\left(\frac{r_2}{r_1}\right)}\ln\left(\frac{r_2}{r_1}\right)$$

$\therefore$ $P = VI$ which proves the relationship power equals the product of the system's voltage and current.

9.4) Characteristic impedance of the transmission line: $Z = \frac{1}{Cv} = \sqrt{\frac{L}{C}}$ (Ω)

& $v \equiv \frac{1}{\sqrt{LC}}$



Gaussian surface: cylindrical prism

$V = \int \vec{E} \cdot \vec{ds}$

$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$

one plate:
$|\vec{E}|(2\pi r^2) = \frac{\sigma(\pi r^2)}{\varepsilon_0}$ where $\sigma = \frac{Q}{A}$

$|\vec{E}| = \frac{\sigma}{2\varepsilon_0}$

two plates: $|\vec{E}| = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

$V = \int \vec{E} \cdot \vec{ds}$

$V = \frac{\sigma}{\varepsilon}\int |ds| \cos(0)$

$V = \frac{\sigma d}{\varepsilon}$

$C = \frac{Q}{\Delta V}$

$C = \frac{\varepsilon Q}{\sigma d}$

$C = \frac{\varepsilon A}{d}$

Plug in $A = w$ & $d = h$:

$C = \frac{\varepsilon w}{h}$

Find inductance L: $\oint \vec{B} \cdot d\vec{l} = \mu I$ (Ampere's Law)

$B = \frac{\mu I}{\omega}$ for uniform magnetic field

$\Phi = \int \vec{B} \cdot d\vec{s} = \mu \frac{dI}{\omega}$

$L = \frac{\Phi}{I} = \mu \frac{d}{\omega}$ } plug in $d = h$

$L = \frac{\mu h}{\omega}$

Characteristic Impedance: $Z = \frac{1}{Cv} = \sqrt{\frac{L}{C}}$ (Equation 8.62)

$$Z = \sqrt{\frac{\mu h}{\omega} \cdot \frac{h}{\varepsilon \omega}} = \sqrt{\frac{\mu h^2}{\omega^2 \varepsilon}} = h\omega\sqrt{\frac{\mu}{\varepsilon}} \rightarrow \boxed{Z = h\omega\sqrt{\frac{\mu}{\varepsilon}}}$$

$$v \equiv \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{\mu h}{\omega} \cdot \frac{\varepsilon \omega}{h}}} = \frac{1}{\sqrt{\mu \varepsilon}} \rightarrow \boxed{v = \frac{1}{\sqrt{\mu \varepsilon}}}$$

In a vacuum, this is the speed of light.
Reminder $\mu \equiv \mu_0 \cdot \mu_r$ & $\varepsilon \equiv \varepsilon_0 \cdot \varepsilon_r$

9.5a) Knowns: $\varepsilon_r = 2.26$, $r_i = 0.406\,mm$, $r_o = 1.48\,mm$

$\Phi = \int \vec{B} \cdot d\vec{A} = 2\int_{r_i}^{r_o} \mu_0 \frac{I}{2\pi r} dr = 2 \frac{\mu_0 I}{2\pi} \ln \frac{r_o}{r_i}$ Note: $\mu_r \sim 1$

$L = \frac{\Phi}{2I} = \frac{\mu_0}{2\pi} \ln \frac{r_o}{r_i}$ $\left(\frac{H}{m}\right)$ (Equation 8.43)

Gauss's Law: $E = \frac{Q}{2\pi \varepsilon r}$

$V = -\int_{r_i}^{r_o} \vec{E} \cdot d\vec{l} = \frac{Q}{2\pi\epsilon} \ln \frac{r_o}{r_i}$

$C = \frac{Q}{V} = \frac{2\pi\epsilon}{\ln\left(\frac{r_o}{r_i}\right)}$ $\left(\frac{F}{m}\right)$ (Equation 8.46)

$$Z = \frac{1}{Cv} = \sqrt{\frac{L}{C}} = \sqrt{\left(\frac{\mu_0}{2\pi} \ln\left(\frac{r_o}{r_i}\right)\right) \cdot \left(\frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi\epsilon}\right)} = \sqrt{\frac{\mu_0}{4\pi^2\epsilon}\left(\ln\left(\frac{r_o}{r_i}\right)\right)^2}$$

$$= \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi} \sqrt{\frac{\mu_0}{\varepsilon}}$$

$$Z = \frac{\ln\left(\frac{r_o}{r_i}\right)}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon}} \qquad \varepsilon = \varepsilon_o \cdot \varepsilon_r$$

Plug in: $Z = \dfrac{\ln\left(\frac{1.48}{0.406}\right)}{2\pi}\sqrt{\dfrac{4\pi\cdot 10^{-7}}{8.854\cdot 10^{-12}(2.26)}} = 51.6\Omega$

$\boxed{Z = 51.6\Omega}$

9.5b) $v = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{\frac{\mu_o}{2\pi}\ln\left(\frac{r_o}{r_i}\right)\frac{2\pi\varepsilon}{\ln\left(\frac{r_o}{r_i}\right)}}} = \dfrac{1}{\sqrt{\mu_o\varepsilon}} = \dfrac{1}{\sqrt{\mu_o\varepsilon_o\varepsilon_r}}$

Plug in: $\dfrac{1}{\sqrt{4\pi\cdot 10^{-7}\cdot 8.854\cdot 10^{-12}\cdot 2.26}} = 1.99\cdot 10^{8}\ m/s$

$\boxed{v = 1.99\cdot 10^{8}\ m/s}$

9.5c) $v = \dfrac{d}{t} \rightarrow d = vt = (1.99\cdot 10^{8})(1\cdot 10^{-9}) = 0.199\ m$

$\boxed{d = 0.199m}$

9.5d) $\left(\dfrac{r_o}{r_i}\right)_{original} = \left(\dfrac{r_o}{r_i}\right)_{proposed} \rightarrow \dfrac{1.48}{0.406} = \dfrac{30}{r_i} \rightarrow \boxed{r_i = 8.23 mil}$

9.5e) $\lambda = \dfrac{v}{f} \rightarrow f = \dfrac{v}{\lambda} = \dfrac{1.99\cdot 10^{8}}{1.48\cdot 10^{-3}} = 134 GHz$

$\boxed{f = 134 GHz}$

9.6a) Known: propagation delay = 4.6 ns/m, $Z = 100\Omega$, ethernet speed: 1Gbps = 10^9 bits/s

frame size = 64 bytes $\rightarrow$ 64 bytes $\cdot \dfrac{8 bits}{1 byte}$ = 512 bits

$v = \dfrac{bits}{t} \rightarrow t = \dfrac{bits}{v} = \dfrac{512\ bits}{10^9\ bits/s} = 512\cdot 10^{-9} s = 512 ns$

$d = \dfrac{t}{propagation\ delay/m} = \dfrac{512 ns}{4.6 ns/m} = 111m \rightarrow \boxed{d = 111m}$

9.6b) reflection coefficient Γ

The source impedance would be mismatched with the effective load impedance by a factor of 2.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25.8 - 51.6}{25.8 + 51.6} = -\frac{1}{3}$$

$\boxed{\Gamma = -\frac{1}{3}}$ $\therefore$ $\frac{1}{3}$ of the signal reflects back due to the impedance mismatch