Antennas Problem Set

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10.1) Equation 9.21: $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$

$$= -\nabla\Phi - i\omega\vec{A}$$

$$= \frac{1}{i\omega\mu_0\varepsilon_0}\nabla(\nabla\cdot\vec{A}) - i\omega\vec{A}$$

Equation 9.25: $\vec{A}(r) = \mu_0 \frac{I_0 d e^{-ikr}}{4\pi r}\hat{z}$

Plugging Eqn 9.25 into 9.21:

$$E = \frac{1}{i\omega\mu_0 E}\nabla\left(\nabla\cdot\left(\mu_0\frac{I d e^{ikr}}{4\pi r}\hat{z}\right)\right) - \frac{\partial\left(\mu_0\frac{I d e^{ikr}}{4\pi r}\hat{z}\right)}{\partial t}$$

10.2) $|P| = \frac{W}{A} = \frac{1000W}{4\pi(10\cdot10^6 m^2)} = 7.96\times10^{-5} W/m^2$

magnitude of Poynting vector 1km from antenna radiating 1kW power

peak electric field strength:

$|P| = \frac{E^2}{R_{rad}} \rightarrow 7.96\times10^{-5}_{W/m^2} = \frac{E^2}{377\Omega}$ → intrinsic impedance of free space

$E = 0.173 V/m \rightarrow E = 173 mV/m$

peak electric field value: $E_{peak} = \sqrt{2}E$

$E_{peak} = 0.245 V/m \rightarrow E_{peak} = 245 mV/m$

10.3) $W_{load} = I^2 R_{load} = \left(\frac{V}{R_{rad}+R_{load}}\right)^2 R_{load}$

$\frac{\partial W}{\partial R_{load}} = -V^2\frac{R_{load}-R_{rad}}{(R_{load}+R_{rad})^3} \longrightarrow$ There's an inflection point at $R_{load} = R_{rad}$.

This is worked out on the following page.

$$\frac{\partial}{\partial R_{load}} \frac{V^2 R_{load}}{(R_{rad}+R_{load})^2} = 0$$ Set derivative = 0.

$$\frac{\partial}{\partial R_{load}}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$ Use quotient rule.

↳ $f' = 1$ & $g = (R_{rad}+R_{load})^2$

$$\frac{(R_{rad}+R_{load})^2 - 2R_{load}(R_{rad}+R_{load})}{(R_{rad}+R_{load})^4} = 0$$ Substitute.

$$R_{rad} + R_{load} - 2R_{load} = 0$$ Factor.

$$R_{rad} - R_{load} = 0$$

$\therefore$ $R_{load} = R_{rad}$ will deliver the maximum power.

10.4) $$G \equiv \max_{\theta,\varphi} \frac{P(r=1,\theta,\varphi)}{W/4\pi}$$ (Equation 9.48)

Gain of antenna ≡ max value of the Poynting vector evaluated on the surface of a unit sphere divided by the total power radiated over that sphere's area.

$k^2 = \frac{\omega^2}{c^2}$ wave equation in free space (part of Equation 9.10)

Plugging in: $$G = 4\pi \frac{\frac{I^2k^2d^2}{32\pi^2 r^2}\sqrt{\frac{\mu}{\varepsilon}}\sin^2\theta}{\frac{I^2k^2d^2}{12\pi}\sqrt{\frac{\mu}{\varepsilon}}} = \frac{\frac{1}{8\pi}}{\frac{1}{12\pi}} = \frac{3}{2} \rightarrow G = \frac{3}{2} = 1.5$$

$A_e = \frac{W_r}{S} = \frac{\text{power received}}{\text{power density of incoming EM wave}}$ → (Poynting vector magnitude)

incident power density: $S \doteq \frac{W_{tot}}{4\pi R^2}$ for isotropic radiator

Plug in $G = \frac{3}{2}$: $$S_r = \frac{G\,W_{tot}}{4\pi R^2} = \frac{\frac{3}{2}W_{tot}}{4\pi R^2} = \frac{3W_{tot}}{8\pi R^2}$$

Solve for W_r (received power): $$W_r = \frac{W_{tot}\,G\,\lambda^2}{(4\pi R)^2} = \frac{W_{tot}\,\frac{3}{2}\,\lambda^2}{(4\pi R)^2}$$

Since $A_e = \frac{W_r}{S}$:

$$A_e = \frac{\frac{W_{tot}\cdot\frac{3}{2}\cdot\lambda^2}{(4\pi R)^2}}{\frac{\frac{3}{2}W_{tot}}{(4\pi R)^2}} = \frac{\frac{3}{2}\lambda^2}{4\pi} = \frac{3\lambda^2}{8} \rightarrow A_e = \frac{3\lambda^2}{8}$$

Ratio of effective area to gain: $$\frac{A_e}{G} = \frac{3\lambda^2}{8\pi}\cdot\frac{2}{3} = \frac{\lambda^2}{4\pi} \rightarrow \frac{A_e}{G} = \frac{\lambda^2}{4\pi}$$