PS12

A → observable.

$\hbar \to h/2\pi$

$[\hat{A}, \hat{H}] = \hat{A}\hat{H} - \hat{H}\hat{A}$

Q1). $\frac{d}{dt}\langle \hat{A} \rangle = \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle$ ← without

$\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle + \left\langle \frac{\delta A}{\delta t} \right\rangle$ → added when the operator A depends on time

Represents direct explicit time variation

Independent of the systems quantum evolution.

$\hat{A}(t)$ changes form over time

accounts for systems own energy dynamic. via hamiltonian (unconserved energy).

Q2) $\hat{\rho}$

$\langle \Psi | \hat{\rho} | \Psi \rangle \geq 0$ for all Ψ

$Tr(\hat{\rho}) = 1$ Let $\{\lambda_i\}$ be eigenvalues of $\hat{\rho}$

$\lambda_i \geq 0$

$\sum \lambda_i = 1$

$Tr(\hat{\rho}^2) = \sum_i \lambda_i^2$

Using cauchy schwarz inequality for real no's

$$\sum_i \lambda_i^2 \leq \left(\sum_i \lambda_i\right)^2 = 1.$$

$$Tr(\hat{\rho}^2) \leq 1.$$

~~Q3~~

$$\hat{\rho} = \sum_i \lambda_i |i\rangle\langle i|$$

$$\hat{\rho}^2 = \left(\sum_i \lambda_i |i\rangle\langle i|\right)^2 \Rightarrow \text{given orthonormality of eigenvectors.}$$

$$= \sum_i \lambda_i^2$$

Q3) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenstate of σ_z, eigenvalue $= 1$.

$\sigma_x \vec{V} = \lambda \vec{V}$

$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ eigenstate of σ_z, eigenvalue $= -1$

$$\det(\sigma_x - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = 0 \quad \boxed{\lambda = \pm 1}$$

for $\lambda = +1$.

$$(\sigma_x - I)V = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow \boxed{a = b}$$

$$|+_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

for $\lambda = -1$

$$(\sigma_x + I)V = 0 \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \boxed{a = -b}$$

$$|-_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \det(\sigma_y - \lambda I) = \lambda^2 - 1 = 0.$$

$$\lambda = \pm 1.$$

for $\lambda = +1$ $$\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - (I)\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \boxed{a = -ib}$$

$$|+_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

for $\lambda = -1$ $$\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + I\right)\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \boxed{a = ib}$$

$$|-y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

3b)
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$$P = |m_{1x} = +1/2\rangle_1 \langle m_{1x} = +1/2|_1$$

from (a)

$$|+\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right) \qquad |-\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x - |-\rangle_x\right).$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\left((|+\rangle_x + |-\rangle_x)_1 (|+\rangle_x - |-\rangle_x)_2 - (|+\rangle_x - |-\rangle_x)_1 (|+\rangle_x + |-\rangle_x)_2\right)\right)$$

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\left(|+\rangle_1|+\rangle_2 - |+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2 - |-\rangle_1|-\rangle_2 - |+\rangle_1|+\rangle_2 - |+\rangle_1|-\rangle_2 \right.$$
$$\left. + |-\rangle_1|+\rangle_2 + |-\rangle_1|-\rangle_2\right).$$

$$= \frac{1}{\sqrt{2}}\left(-|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2\right).$$

$P = |+\rangle_1 \langle +|_1$ project first spin onto $|+\rangle_x$

$$P(|\Psi\rangle) = \frac{1}{\sqrt{2}}\left(-|+\rangle_1 \langle +|+\rangle_1 |-\rangle_2 + |+\rangle_1 \langle +|-\rangle_1 |+\rangle_2\right).$$

($\langle +|+\rangle_1 \to 1$, $\langle +|-\rangle_1 \to 0$)

$$P|\Psi\rangle = -\frac{1}{\sqrt{2}}\left(|+\rangle_1 |-\rangle_2\right).$$

$|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ from ⓐ

on y

$m_{1y} = +1/2$

$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_y + |-\rangle_y\right)$ $\quad |\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle_y - |-\rangle_y\right).$

Using same steps as above. (in my notebook)

$$\Psi = \frac{1}{\sqrt{2}}\left(-|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2\right).$$

$P = |+\rangle_1 \langle +|_1$

$P(\Psi) \longrightarrow$ second spin state $|-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.$

3c) $\hat{S}_x = \frac{\hbar \sigma_x}{2}$ $\quad \hat{S}_y = \frac{\hbar}{2}\sigma_y.$

$$\langle \Psi | \hat{S}_{1x} \hat{S}_{2y} | \Psi \rangle = \left(\frac{\hbar}{2}\right)^2 \langle \Psi | \sigma_{1x} \sigma_{2y} | \Psi \rangle$$

$$\langle \Psi | S_{1y} S_{2x} | \Psi \rangle = \frac{\hbar^2}{4} \langle \Psi | \sigma_{1y} \sigma_{2x} | \Psi \rangle.$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right)$$

$$\langle \Psi | \sigma_i \sigma_j | \Psi \rangle = -\delta_{ij}$$

$= 0$ if $i \neq j$

for two different directions.

$$\sigma_{1x} \sigma_{2y} = \sigma_x \otimes \sigma_y$$

$$\sigma_{1y} \sigma_{2x} = \sigma_y \otimes \sigma_x.$$

$$\Rightarrow \langle \Psi | \sigma_{1x} \sigma_{2y} | \Psi \rangle = - \langle \Psi | \sigma_{1y} \sigma_{2x} | \Psi \rangle.$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle - |10\rangle \right)$$

$= -i$ $\quad$ $= +i$

hence proven.

3d). $m_{1x} m_{2y} = - m_{1y} m_{2x}$

$[\sigma_{1x}, \sigma_{2y}] = 0$.

since two different particles

$$\langle \Psi | \sigma_{1x} \sigma_{2y} | \Psi \rangle = - \langle \Psi | \sigma_{1y} \sigma_{2x} | \Psi \rangle$$

proved

3e). $\langle \Psi | \hat{\sigma}_{1x} \hat{\sigma}_{1y} + \hat{\sigma}_{1y} \hat{\sigma}_{2x} | \Psi \rangle$ $\Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \right).$

$$\Rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

using $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$$\sigma_x \otimes \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

$$\sigma_y \otimes \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

$$\sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \end{pmatrix}.$$

now $\langle \Psi | (\sigma_{1x}\sigma_{2y} + \sigma_{1y}\sigma_{2x}) | \Psi \rangle$

$$= \begin{pmatrix} 0 & 0 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \boxed{0}$$

3f) In d concluded sum of cross terms vanished.

In e Same result from matrix computation in z base.

Q4) $R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\vec{\sigma}}$

$A = \sum_i \lambda_i P_i$ and $f(A) = \sum_i f(\lambda_i) P_i$

← hermitian operator

$n \cdot \sigma$ has eigenvalues ± 1

let P_+, P_- be projectors onto the eigenstate

$$e^{-i\frac{\theta}{2}\hat{n}\sigma} = e^{-i\frac{\theta}{2}}(P_+) + e^{i\theta/2}(P_-)$$

$P_+ = \frac{1}{2}(I + \hat{n}\vec{\sigma})$ $\quad P_- = \frac{1}{2}(I - \hat{n}\vec{\sigma})$

$$\Rightarrow R_{\hat{n}}(\theta) = e^{-i\theta/2}\left(\frac{1}{2}(I + \hat{n}\vec{\sigma})\right) + e^{i\theta/2}\left(\frac{1}{2}(I - n\sigma)\right).$$

$$= \frac{1}{2}\left((e^{-i\theta/2} + e^{i\theta/2}) I + (e^{-i\theta/2} - e^{i\theta/2})\hat{n}\hat{\sigma}\right)$$

$$= \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\hat{n}\vec{\sigma}$$

Q5>

$U^2 = X$ → NOT.

$\sqrt{NOT}$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\sqrt{x}$ $\Rightarrow$ $U^2 = \frac{1}{4}\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

$$\therefore U = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Q6> $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

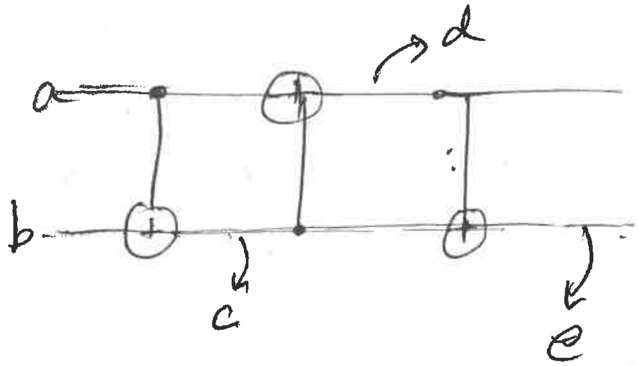
$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Initial State $|000\cdots 0\rangle^{N}$

$H \otimes |00\cdots\cdots 0\rangle$ $\quad H^{\otimes N}|0\rangle^{\otimes N} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)^{\otimes N}$

$$= \frac{1}{(\sqrt{2})^N} \sum_{x \in \{0,1\}^N} |x\rangle$$

Q7)



$c = b \oplus a$

$\underline{d} = a \oplus c$

$\underline{e} = a \oplus d$

two output

$d = a \oplus (b \oplus a)$

$e = (b \oplus a) \oplus (a \oplus (b \oplus a))$

Simplify.

$\boxed{d = b}$ $\Rightarrow$ $e = (b \oplus a) \oplus b$

$\boxed{e = a}$

Swaps the two. using entanglement.

Q8). QFT