PS3

Tuesday, February 25, 2025 1:15 PM

Q1) a) Binomial probability $P_n(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

Stirlings approximation $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$N = np$

$$\Rightarrow P_n(x) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

for small y

$\ln(1+y) =$ $\Rightarrow$ $1+y = e^y$

$= e^y 1$

for $-y$ $\Rightarrow$ $1-y = e^{-y}$

replace with p

$n \times p = N$

$p = (N/n)$ $\Rightarrow$ $$P_n(x) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{x!\,(n-x)!}\left(\frac{N}{n}\right)^x \left(e^{-p}\right)^{n-x}$$

$$= \frac{\sqrt{2\pi n}\; n^{n-x}\; N^x\; e^{-n(p+1)+xp}}{x!\,(n-x)!}$$

for very high 'n'

$n! \approx (n-x)!\, n^x$ for large n

$$P_{n\to\infty}(x) = \frac{\cancel{\sqrt{2\pi n}}\; n^{\cancel{n}-x}\; N^x\; e^{-np-\cancel{n}}\; e^{xp}}{x!\;\cancel{\sqrt{2\pi n}}\left(\frac{\cancel{n}}{\cancel{e}}\right)^{\cancel{n}} n^{-x}}$$

$$= \frac{(N/\cancel{n})^x\, e^{-np}}{x!\;\cancel{n^{-x}}}\; e^{\enclose{circle}{xp}} \rightarrow 0$$

$$P_{n\to\infty}(x) = \frac{N^x e^{-N}}{x!}$$

b) $\langle x(x-1)\cdots(x-m+1)\rangle = N^m$

$$= \sum_{x=m}^{\infty}[x(x-1)\cdots(x-m+1)]\,p(x)$$

$p(x) = \frac{N^x e^{-N}}{x!}$

$$= \sum_{x=m}^{\infty}\underbrace{[x(x-1)\cdots(x-m+1)]}\frac{e^{-N}N^x}{x!}$$

$$= e^{-N}\sum_{x=m}^{\infty}N^x \times \frac{1}{(x-m)!}$$

$$= e^{-N}N^m\underbrace{\sum_{x=m}^{\infty}\frac{N^{x-m}}{(x-m)!}}$$

for a poisson distr.

$e^{-N}\sum_{0}^{\infty}\frac{N^x}{x!} = 1$

since the series is from $m \to \infty$

shifted

$$\downarrow$$

$$= \boxed{N^m \times 1}$$

c) $\frac{\sigma}{\langle x\rangle} = \frac{1}{\sqrt{N}}$

in above qn.

$\langle x\rangle = N$

$\langle x^2\rangle = N^2 + N$

$\langle x^2 - x\rangle = N^2$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$$

$$= N^2 + N - (N)^2$$

$$\sigma = \sqrt{N}$$

$$\frac{\sigma}{\langle x\rangle} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}$$

5.2) Counting follows poisson statistics

←

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$$\frac{\sigma}{N} = \frac{1}{\sqrt{N}}$$

$$\frac{\text{variance}}{\text{photon}} = 1\% = \frac{1}{100}$$

$$\underline{N = 10^4}$$

for 1 ppm $N = (10^6)^2 = \underline{10^{12}}$

per photons $E = hf$

$f \approx 5\times10^4 Hz$ $h = 6.626\times10^{-34} Js$

$\approx 3.3\times10^{-17} J$

$P = NE$

for $10^4 \longrightarrow$ $\underline{P \approx 3.3\times10^{-15} W}$

$10^{12} \longrightarrow$ $\underline{P \approx 3.3\times10^{-7} W}$.

Q5.3) a) Johnsons source noise $\rightarrow \sqrt{4kTR\Delta f}$

$= \sqrt{4\times k\times 300\times 10K\times 20kHz}$

source imp

bw.

$= 1.8\times10^{-6} V_{rms}$.

for SNR = 20dB

$= 20\log\frac{V_s}{V_N} \Rightarrow$ $V_s = 10 V_N$

$\underline{V_s = 18\mu V}$

b) $\frac{1}{2}kT = \frac{1}{2}CV^2$

Thermal noise $\rightarrow V_{nois}$

$* \ V = \sqrt{KT}$

$\frac{1.3\times10^{-23}\times 300}{}$

Thermal noise → V_{nois}

$$\Rrightarrow V_{nois} = \sqrt{\frac{KT}{C}}$$

$$\frac{1.3 \times 10^{-23} \times 300}{18^2 \times 10^{-2}}$$

$$C = \frac{KT}{V_{nois}^2} \Rightarrow \boxed{1.2nF}$$

current

c) Shot noise

$$\hookrightarrow I_{nois} = \sqrt{2qI\Delta f}$$

$$(0.01I)^2 = 2qI\Delta f \quad //$$

$$I = \frac{2q\Delta f}{(0.01)^2}$$

$q = \underline{1.6 \times 10^{-19}}$ e charge.

$$I = \frac{2q(20000)}{10^{-4}} \boxed{= 64pA}$$

Q5.4)

a) $P_0(t) \longrightarrow x=0$ at t

$P_1(t) \longrightarrow x=1$ at t

x

$0 \longrightarrow 1$ in $dt \longrightarrow \alpha\, dt$

$1 \longrightarrow 0$ in $dt \longrightarrow \beta\, dt$

$$\frac{dP_0}{dt} = -\alpha P_0 + \beta P_1$$

$$\frac{dP_1}{dt} = \alpha P_0 - \beta P_1$$

$$\frac{d}{dt}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix} \quad //$$

b) - first finding eigen values: λ

$$\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix} - \lambda I = 0 \Rightarrow \begin{vmatrix} -\alpha-\lambda & \beta \\ \alpha & -\beta-\lambda \end{vmatrix} = 0.$$

$$(\alpha+\lambda)(\beta+\lambda)-\alpha\beta = 0$$

$$\lambda(\lambda+\alpha+\beta) = 0$$

$\lambda = 0$
so
$-(\alpha+\beta)$.

Now to find the eigen vectors. $\rightarrow \left(\begin{bmatrix}-\alpha & \beta \\ \alpha & -\beta\end{bmatrix} - \lambda I\right) eig = 0$

1) for $\lambda = 0$

$$\begin{bmatrix}-\alpha & \beta \\ \alpha & -\beta\end{bmatrix}\begin{bmatrix}e_1 \\ e_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$-\alpha e_1 + \beta e_2 = 0$
$\alpha e_1 - \beta e_2 = 0$.

$e_1 = \beta$, $e_2 = \alpha$

$$\text{eig1} = \begin{bmatrix}\beta \\ \alpha\end{bmatrix}$$

2) for $\lambda = -(\alpha+\beta)$

$$\begin{bmatrix}-\alpha+\alpha+\beta & \beta \\ \alpha & -\beta+\alpha+\beta\end{bmatrix}\begin{bmatrix}e_1' \\ e_2'\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$\beta e_2' + \beta e_1' = 0$.
$\alpha e_1' + \alpha e_2' = 0$

$$\text{eig2} = \begin{bmatrix}1 \\ -1\end{bmatrix}$$

eig $P = \begin{bmatrix}\beta & 1 \\ \alpha & -1\end{bmatrix}$ eis det(P) / or $|P| = |-\beta-\alpha|$ $\alpha > 0$, $\beta > 0$

so invertible.

eig $P^{-1} = \dfrac{Adj(P)}{-(\alpha+\beta)}$

$Adj(P) = \begin{bmatrix}-1 & -1 \\ -\alpha & +\beta\end{bmatrix}$

$D = P^{-1} A P$

$$= \frac{-1}{(\alpha+\beta)}\begin{bmatrix}-1 & -1 \\ -\alpha & \beta\end{bmatrix}\begin{bmatrix}-\alpha & \beta \\ \alpha & -\beta\end{bmatrix}\begin{bmatrix}\beta & 1 \\ \alpha & -1\end{bmatrix} = \frac{-1}{(\alpha+\beta)}\begin{bmatrix}-1 & -1 \\ -\alpha & \beta\end{bmatrix}\begin{bmatrix}0 & \cancel{\alpha}\beta \\ 0 & \alpha+\beta\end{bmatrix}$$

$$= \frac{-1}{(\alpha+\beta)}\begin{bmatrix}0 & 0 \\ 0 & (\alpha+\beta)^2\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & -(\alpha+\beta)\end{bmatrix}$$

$\rightarrow$ d $\begin{bmatrix}P_0'\end{bmatrix}$ $\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}P_0'\end{bmatrix}$ $\rightarrow$ transformed:

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}p_0'\\ p_1'\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & -(\alpha+\beta)\end{bmatrix}\begin{bmatrix}p_0'\\ p_1'\end{bmatrix}$$

① → transformed.

$$\Rightarrow \frac{dp_0'}{dt} = 0 \qquad \underline{p_0' = c_1} \quad ✓$$

$$\frac{dp_1'}{dt} = -(\alpha+\beta)p_1' \Rightarrow \underline{p_1' = c_2 e^{-(\alpha+\beta)t}} \quad ✓$$

$$p' = \begin{bmatrix}c_1\\ c_2 e^{-(\alpha+\beta)t}\end{bmatrix} \qquad p = \text{eig}_p \times p'$$

$$= \begin{bmatrix}\beta & +1\\ \alpha & -1\end{bmatrix}\begin{bmatrix}c_1\\ c_2 e^{-(\alpha+\beta)t}\end{bmatrix}$$

$$\begin{matrix}p_0 \leftarrow \\ p_1 \leftarrow\end{matrix} \quad = \underline{\begin{bmatrix}\beta c_1 + c_2 e^{-(\alpha+\beta)t}\\ \alpha c_1 - c_2 e^{-(\alpha+\beta)t}\end{bmatrix}}$$

Initial cond.

$p_{0\,at\,0} = p_0^0$ ✓

$p_{1\,at\,0} = 1 - p_0^0$ ✓

$$\longrightarrow \left.\begin{matrix}\beta c_1 + c_2(1) = p_0^0\\ \alpha c_1 - c_2(1) = 1 - p_0^0\end{matrix}\right] \qquad \begin{matrix}(\alpha+\beta)c_1 = 1\\ c_1 = 1/\alpha+\beta\end{matrix}$$

$$c_2 = p_0^0 - \frac{\beta}{\alpha+\beta}$$

c) Same as my method; <u>check notebook</u> → too long