Problem Set 4: Nature of Mathematical Modelling

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1 Problem1: Emergence of thermodynamics

1.1 Analytical solution of Caldeira-Leggett Model

Caldeira-Leggett (CL) model:

$$\frac{dx_k}{dt} = p_k/m_k \qquad \text{for k=1,...,N}$$

$$\frac{dp_0}{dt} = -m_0\Omega^2 x_0 + \sum_{k=1}^{N} g_k(w_k x_k - g_k x_0/m_k)$$

$$\frac{dp_k}{dt} = -m_k w_k^2 x_k + g_k w_k x_0 \qquad \text{for k=1,...,N}$$

Let vector $\overrightarrow{\mathbf{Z}} = (x_0, x_1, ..., x_N, p_0, p_1, ..., p_N)^T$. The above equations can be written in matrix form:

$$\frac{d\overrightarrow{\mathbf{Z}}}{dt} = A\overrightarrow{\mathbf{Z}}$$

$$A = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

$$\text{where} \quad B = \begin{bmatrix} 1/m_0 & 0 & 0 & \dots & 0 \\ 0 & 1/m_1 & 0 & \dots & 0 \\ 0 & 0 & 1/m_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1/m_N \end{bmatrix}$$

$$C = \begin{bmatrix} -m_0\Omega^2 - \alpha & g_1\omega_1 & g_2\omega_2 & \dots & g_N\omega_N \\ g_1\omega_1 & -m_1\omega_1^2 & 0 & \dots & 0 \\ g_2\omega_2 & 0 & -m_2\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ g_N\omega_N & 0 & 0 & \dots & -m_N\omega_N^2 \end{bmatrix} \qquad \alpha = \sum_{k=1}^{N} g_k^2/m_k$$

Now we calculate the eigenvalues of A.

$$\begin{vmatrix} \lambda I & -B \\ -C & \lambda I \end{vmatrix} = |\lambda I||\lambda I - CB/\lambda| = |\lambda I||\frac{1}{\lambda}(\lambda^2 I - CB)|$$

Therefore, the square of eigenvalues of A are the eigenvalues of CB.

$$\text{CB has the form:} \quad \begin{bmatrix} -\Omega^2 - \alpha/m_0 & g_1\omega_1/m_1 & g_2\omega_2/m_2 & \dots & g_N\omega_N/m_N \\ g_1\omega_1/m_0 & -\omega_1^2 & 0 & \dots & 0 \\ g_2\omega_2/m_0 & 0 & -\omega_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ g_N\omega_N/m_0 & 0 & 0 & \dots & -\omega_N^2 \end{bmatrix}$$

The square root of each eigenvalue of CB and its conjugate form one pair of the eigenvalues of A. The eigenvalues of CB can be solved by partitioning the matrix as follows:

$$
\begin{aligned}
A_{11} &= -\,\Omega^2 - \alpha/m_0; \\
A_{12} &= \begin{bmatrix} g_1\omega_1/m_1 & \ldots & g_N\omega_N/m_N \end{bmatrix}; \\
A_{21} &= \begin{bmatrix} g_1\omega_1/m_0 \\ \ldots \\ g_N\omega_N/m_0 \end{bmatrix} \\
A_{22} &= diag(-\omega_1^2, ..., -\omega_N^2)
\end{aligned}
$$

If $\lambda I - A_{22}|$ is invertible,

$$|\lambda I - CB| = \begin{vmatrix} \lambda - A_{11} & -A_{12} \\ -A_{21} & \lambda I - A_{22} \end{vmatrix} = |(\lambda - A_{11}) - A_{12}(\lambda I - A_{22})^{-1}A_{12}||\lambda I - A_{22}|$$

The characteristic polynomial is

$$\lambda + \Omega^2 + \alpha/m_0 - [\frac{(g_1\omega_1)^2/m_0 m_1}{\lambda + \omega_1^2} + \frac{(g_2\omega_2)^2/m_0 m_2}{\lambda + \omega_2^2} + ... + \frac{(g_N\omega_N)^2/m_0 m_N}{\lambda + \omega_N^2}] = 0$$

1.2 Numerical example

Let N=200, $\Omega = 1$, $\gamma = 1$, $m_k = 1$, $\omega_k = 10k/N$, and $g_k = \sqrt{(}40\gamma/(N\pi))$. The Matlab script is following: `C1 = eye(201);`\\for i=2:201
C1(i,i)=-((i-1)/20)^2;
end
$C1(1,1) = -1 - 40/pi;$

C2 = zeros(201);
for i=2:201
C2(1,i)=(i-1)/20;
C2(i,1)=(i-1)/20;
end

C=C1+C2; B=eye(201); A=zeros(402); A(1:201,202:402)=B;
A(202:402,1:201)=C; [V,D]=eig(A); y=0;
for i=1:201
if(real(D(i,i))¿0imag(D(i,i)) =0)
D(i,i)
y=y+1;

end
end
y
It's oscillatory with 51 modes.

1.3 Plot the result

1.4 Another system

$$\frac{d}{dt}\begin{bmatrix} Q \\ P \end{bmatrix} = \begin{bmatrix} 0 & 1/m_0 \\ -m_0(\Omega^2+\gamma^2) & -2\gamma \end{bmatrix}\begin{bmatrix} Q \\ P \end{bmatrix}$$

The eigenvalues of the center matrix is easy to calculate:

$$\begin{vmatrix} \lambda & -1/m_0 \\ m_0(\Omega^2+\gamma^2) & \lambda+2\gamma \end{vmatrix} = 0$$

The quadratic equation $\lambda^2+2\gamma\lambda+(\Omega^2+\gamma^2)=0$ has two conjugate complex roots: $-\gamma+\Omega i$. Therefore, the system is damped oscillation. Eventually the oscillation dies out.

2 Problem 2: SVD and PCA

2.1 Derivation of SVD

If $\mathbf{A}$ is a matrix of size $m\times n$ then there exists orthogonal matrices $\mathbf{V}(m\times m)$ and $\mathbf{W}(n\times n)$ such that $\mathbf{V^T A W} = diag(\sigma_1,...,\sigma_p)$
with $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_p \geq 0$, $p=\min(m,n)$.
Proof
Since $\mathbf{A^T A} \succeq 0$, the eigenvalues of $\mathbf{A^T A}$ are nonnegative $\sigma_1^2 \geq \sigma_2^2 \geq ... \geq \sigma_n^2 \geq 0$. Let all the r nonnegative σ_i be sorted as $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq 0$.
Let x_k be norm 1 eigenvectors of $\mathbf{A^T A}$ corresponding to σ_k^2 for $k=1,2,...,r$.
Let $y_k = \mathbf{A}x_k/\sigma_k$. Notes show that y_k is orthonormal.
Construct $\mathbf{W} \equiv \begin{bmatrix} x_1 & ... & x_n \end{bmatrix}$ and $\mathbf{V} \equiv \begin{bmatrix} y_1 & ... & y_r & \mathbf{V_2} \end{bmatrix}$. Then

$$\mathbf{V^T} = \begin{bmatrix} y_1^T \\ y_2^T \\ \cdots \\ y_r^T \\ \mathbf{V_2^T} \end{bmatrix}$$

3 Problem 3: Resistor Network

3.1 the matrix constraints for voltages and current

The node voltages and branch currents are defined as following:

1. voltages are already marked on the network as$V_1, V_2, ..., V_8, V_a, V_b, ..., V_h, V_{out}$;
2. branch current $I_i$flows into node with voltage $V_i, i = 1, .., 8$;
3. branch current $I_a$flows out of node with voltage V_a,etc.
4. I_{out} flows out of node with voltage V_{out}.

Matrix constraints on voltages has the form: $\mathbf{GV} = \mathbf{I}$.
Voltage vector is: $V = [V_1, V_2, ..., V_8, V_a, ..., V_h, V_{out}]^T_{17\times 1}$;
Current vector is: $I = [I_1, I_2, ..., I_8, I_a, ..., I_h, I_out]^T_{17\times 1}$;
G is partitioned as:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

$$G_{11} = -g_1 \mathbf{I_{8\times 8}}, \quad G_{12} = g_1 \mathbf{I_{8\times 8}}, G_{21} = \mathbf{0_{9\times 8}}$$

$$G_{22} = \begin{bmatrix} g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -g_2 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -g_2 & g_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_2 & g_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g_2 & g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_2 & g_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -g_2 & g_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g_2 & g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_2 & g_2 \end{bmatrix}$$

The current matrix is $\mathbf{KI} = \mathbf{0}$, where $\mathbf{K} = \begin{bmatrix} \mathbf{I_{8\times 8}} & K_{12} \end{bmatrix}$.

$$K_{12} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$